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ABSTRACT

This material is designed to examine the research on how children acquire basic addition and subtraction concepts and skills. Two major lines of theories of the development of basic number concepts, called logical concept and quantification skill approaches, are identified. Major recurring issues in the development of early number concepts are also discussed. This is followed by examination of studies specifically related to addition and subtraction. It is noted that by kindergarten most children understand the rudiments of these operations, but a complete understanding develops over a protracted span of years. The direction of initial and ongoing research is discussed, and word problem studies and investigations on children's solutions of symbolic addition and subtraction problems are viewed to be concerned with both problem difficulty and solution processes. The review suggests that a great deal is known both about children's knowledge of addition and subtraction and how they solve addition and subtraction problems. It also indicates that there is not yet a clear picture of how to apply such new information to design more effective instruction. (MP)

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A Research on Addition and Subtraction

Thomas P. Carpenter, Sheldon Blume, James Hiebert, Constance Martin Anick, and David Pimm

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March 1980

Working Paper No. 330

A REVIEW OF RESEARCH ON ADDITION AND SUBTRACTION

by

Thomas P. Carpenter, Glendon Blume,
James Hiebert, Constance Martin Anick,
David Pimm

Report from the Program on
Student Diversity and Classroom Processes

Wisconsin Center for Education Research
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Abstract

The purpose of this paper is to review the research on how children acquire basic addition and subtraction concepts and skills. The review starts with an analysis of major theories of the development of basic number concepts. Two major lines of theories are identified: logical concept theories and quantification skill theories. The most notable example of a logical concept theory is represented by the work of Piaget while the work of Gelman and Klahr and Wallace provide the most comprehensive examples of quantification skill theories. Ginsburg's work attempts to build a bridge between the two approaches. Major recurring issues in the development of early number concepts are also discussed. These include the relation between counting and subitizing, how number conservation develops, and how early number operations affect the acquisition of addition and subtraction skills.

The second section of the paper examines research on basic concepts specifically related to addition and subtraction. By kindergarten most children understand that joining an element to a set increases its numerosity and removing an element decreases its numerosity. However, a complete understanding of basic properties of addition and subtraction, like inversion and compensation, develop over a protracted span of years.

Most early research on children's solutions of symbolic addition and subtraction problems was limited to problem difficulty. Much of the initial research focused on the relative difficulty of different basic number facts. More recently, the emphasis has shifted to an analysis of children's solution processes. Using a variety of paradigms, a number of solution strategies based on counting have been identified.

Research on word problems has also been concerned with both problem difficulty and solution processes. Most of the problem difficulty research has focused on syntax variables. Current research on children's solution processes has attempted to make a connection between the semantic structure of problems and the processes children use to solve them. A mounting body of evidence clearly demonstrates that young children can solve a variety of addition and subtraction problems by physically modeling the action or relationships described in the problems. These direct modeling strategies gradually give way to more abstract counting representations that retain certain characteristics of the initial modeling strategies. Several simulation models have been developed to account for the internal cognitive mechanisms that are required for the observed patterns of development.

The review suggests that a great deal is known about both children's knowledge of addition and subtraction and how they solve addition and subtraction problems, but also indicates that there is not yet a clear picture of how to apply these insights to design more effective instruction.

INTRODUCTION

The learning of basic addition and subtraction concepts and skills is a major objective of primary school mathematics instruction. Because of the central place it occupies in the mathematics curriculum, there is an extensive body of research on the teaching and learning of addition and subtraction that dates back to the turn of the century. In the last few years, there has been a resurgence of interest in the study of addition and subtraction as researchers have found that the techniques of cognitive psychology provide new insights into the processes that children use in solving addition and subtraction problems.

The purpose of this paper is to review this growing body of research on addition and subtraction. The paper is divided into six major parts. To provide some background, the first section briefly summarizes some of the major lines of inquiry on the development of basic number concepts. The second section deals with children's understanding of basic concepts underlying addition and subtraction. The third section focuses on addition and subtraction problems presented in a symbolic context, and the fourth section examines research on word problems. The fifth section attempts to synthesize what is known about the general pattern of development of addition and subtraction processes, and the sixth section provides a general discussion of issues and areas of needed research. The paper limits its review to research on addition and subtraction with relatively small numbers. Problems that require the use of algorithms depend on the development of place value concepts and are beyond the scope of this review.

THE DEVELOPMENT OF BASIC NUMBER CONCEPTS

In recent years, there has been an increasing interest in the development of early number concepts by psychologists. Within the past few years alone several major works have been published on the subject (Brainerd, 1979; Gelman & Gallistel, 1978; Ginsburg, 1977a; Klahr & Wallace, 1976). There are several reasons for this. First, the developing ability to deal with quantitative concepts has come to be recognized as a milestone in children's intellectual development (Klahr & Wallace, 1976; Piaget, 1952). Consequently, the acquisition of number concepts provides for the psychologist a legitimate area of study in its own right. Second, the way in which young children learn to deal with quantitative situations serves as a window to other dimensions of cognitive development (Gelman & Gallistel, 1978). Thus, number concepts provide a vehicle from which to study the growth of the intellect in general.

Due to this prodigious input from developmental psychology, the initial section of this paper will draw heavily from psychologically oriented research. The aim of this section is to outline the major approaches which have been taken in investigating the development of early number concepts. The purpose is to present the fundamental theoretical positions which have guided empirical activities in this area rather than to review all relevant research. The discussion focuses on the development of number concepts prior to symbolization. It deals with children's ability to quantify sets and to reason about them. However, it stops short of describing the development of operations

with quantities in the arithmetical sense. Borrowing Elkind's (1969) distinction, this section deals with operations within sets, like quantifying and conserving, but not with operations between sets, like addition and subtraction. It attempts to outline how the development of basic number concepts might influence the acquisition of arithmetic operations but leaves the description of the acquisition process itself to later sections of the paper.

Many investigators have studied the development of number concepts, each from a somewhat different perspective. However some important similarities can be detected. For example, all have used a developmental approach. That is, they have focused on the way in which young children's conception of, and skill with, number changes over time. In describing this development, most investigators have identified two major components of children's proficiency with number--ability to quantify or assign number to specific sets, and ability to reason about number. While most researchers recognize the importance of these two abilities, they do not all agree on their developmental sequence, i.e., the order in which these abilities are acquired.

This disagreement represents a fundamental difference between the major positions on the growth of number concepts, a difference which serves to partition theory and research into two distinct camps. One assumes that children must acquire certain logical reasoning abilities before they can apply quantification processes in any meaningful way, while the other argues that children can reason about number only after they have quantified sets and have specific numerosities in mind. Some

of the differences between these positions can be resolved by pointing to differences in definitions of "number" and differences in tasks used to assess number concepts. However some basic disparities still remain. The purpose of the following discussion is to outline these two major positions on the development of early number concepts by reviewing the work of each position's primary proponents.

LOGICAL CONCEPT THEORIES

Several investigators have adopted the position that number is the outgrowth of more basic logical concepts. Therefore, the development of number concepts are believed to depend upon the development of certain logical reasoning abilities. Foremost among the advocates of this position is Piaget (1952). So great has been Piaget's influence in this area of research that Flavell (1970) concluded, "Virtually everything of interest that we know about the early growth of number concepts grows out of Piaget's pioneer work in the area" (p. 1001).

To understand Piaget's (1952) view of number concept development, it is useful to review a critical distinction which Piaget (1964, 1970) made between logical-mathematical knowledge and physical knowledge. The first type of knowledge is generated by internal mental processes while the second is achieved by direct contact with the external environment via sensory perceptions (Steffe, 1976). The first arises from deduction and is verifiable by logical reasoning; the second arises from induction and is verifiable by empirical test (Beilin, 1976). The first generalizes across content, transfers to related problems, and is reconstructable (i.e., is not based on recall); the second is content-specific and is subject to memory loss (Furth, 1969).

For Piaget (1952; see also Beth & Piaget, 1966), number is a form of logical-mathematical knowledge. As such, the construction of number is thought to be closely tied to the development of logic. In particular, number depends upon the logic of classes and asymmetrical relations. Classes are collections of objects grouped together on the basis of a common quality or attribute, and classification leads to hierarchies of classes and the notion of part-whole. Relations refer to ways in which the objects within a class may be compared. The primary activity here is seriation (arranging the objects in order according to some continuous attribute) which is based on the asymmetrical relation of transitivity. The logic of classes corresponds to the cardinal aspects of number while the logic of relations corresponds to its ordinal aspects.

According to Piaget, children initially consider the logical activities of classification and seriation as separate and independent entities resulting in a dual system of logic. However, they are eventually fused into a single system, and a primary result of this fusion or synthesis is the concept of number.

A hallmark of children's growing understanding of number is the ability to conserve (i.e., to mentally preserve one-to-one correspondence). The principle of conservation is of critical importance in Piaget's theory and represents the segment of Piaget's work which undoubtedly has had the greatest impact on subsequent research. Piaget (1952) left no doubt about the role he saw conservation to play:

> Our contention is merely that conservation is a necessary condition for all rational activity. . . . A set or collection is only conceivable if it

> remains unchanged irrespective of the changes occurring in the relationship between the elements. . . . Number is only intelligible if it remains identical with itself, whatever the distribution of the units of which it is composed. (pp. 3-4)

He has described a stagewise development of number concepts in which conservation, seriation, and classification develop in close synchrony. In the first stage, children are dominated by immediate perceptual qualities of an event and give little evidence of logical reasoning. Only gross quantification, absolute quantifying ideas such as "more" and "less," are evident, and these are based on perceptual judgments. If equivalence is not perceived it is thought not to exist. As a consequence, children in the first stage do not conserve, are incapable of seriation, and do not understand simple class inclusion relationships.

Stage two is a transitional period. Some progress is made on all fronts so that children can construct series and one-to-one correspondences. But they still have difficulty when either is spatially transformed. Quantification at this stage is "intensive." That is, quantities cannot yet be combined in the numerical sense but only compared in terms of "bigger than" or "smaller than," based on perception. Solutions come by empirical substantiation rather than by logical necessity.

The third stage brings a series of major breakthroughs in the child's thought. The focus on perceptual cues and on the qualities of objects, which has dominated in the past, now shifts to quantitative aspects. Up to this point, the objects in a class have been distinguished on the

basis of perceived qualities. Now, however, children are able to suppress these qualities and the individual elements of a class are seen to be equivalent in all relevant respects. Their only distinction from one another is their relative position, or order, which is imposed by the child's seriation. This is the notion of unit, the basic numerical concept. All elements are at the same time equivalent (belonging to the same class) and different (by virtue of their enumerated position).

Out of these developments emerge the two complementary aspects of number--cardinality and ordinality. Cardinality refers to the numerousness of a class; all classes which can be put into one-to-one correspondence have the same cardinal value. With the newly acquired ability to decompose a class into units, the child can now conserve and can understand class inclusion relationships. Ordinality is also fully understood. The decomposition of a set into units entails the realization that an object's position completely defines the object.

Along with the achievement of cardinal and ordinal understanding comes an immediate synthesis or fusion to form the complete number concept. The child can now identify the ordinal position of an object with the sum of that object and those preceding it, i.e., its cardinal value. Once again, Piaget (1952) commented about the close relationship between number and logical reasoning, "the psychological, as well as the logical, constitution of classes, relations, and numbers is a single development, whose respective changes are synchronic and interdependent" (p. 157).

In summary, Piaget viewed number as a logical-mathematical concept which is constructed by the child, rather than a physical concept which

is discovered by the child through sensory perceptions. By definition, an understanding of number requires an understanding of conservation, class inclusion, and seriation. While Piaget acknowledged that certain quantifying skills, such as counting, are acquired prior to the full development of these logical reasoning abilities, he contended that they take on meaning only with the onset of logical thought. Evidence cited for this position comes from the observation that these quantifying skills do not help young children to solve the logical reasoning tasks. Thus, the early acquisition of quantification skills is believed to make no significant contribution to the development of a mature number concept.

Other investigators have adopted positions similar to Piaget's with regard to the developmental relationship between logical reasoning abilities and quantification skills, but differ with Piaget, either with respect to their focus of interest or with respect to their logical analysis of the number concept. Brainerd (1973a, 1973c, 1976, 1979) disagreed with Piaget on the developmental sequence of the logical notions underlying number. Piaget contended that an operational understanding of number results from the concurrent development of cardinal and ordinal concepts. In contrast, Brainerd (1979) believes the ordinal concept to be a more desireable logical foundation for number, and ordinal number to be psychologically more basic than cardinal number. While there has been considerable debate about what logical foundation for number is most consistent with psychological reality (Beth & Piaget, 1966; Brainerd, 1973a, 1979; Macnamara, 1975, 1976; Piaget, 1952), Brainerd maintained that ordinal number concepts emerge earlier than cardinal concepts and that ordinal

number plays a more important role in the early growth of arithmetic concepts. Brainerd proposed the following developmental sequence: ordinal number-natural number-cardinal number.

While the results of Brainerd's research (Brainerd, 1973b, 1974, 1977; Brainerd & Fraser, 1975; Brainerd & Kaszor, 1974) supported the ordinal-cardinal sequence, there are several limitations which dissuade drawing broad theoretical conclusions from the findings. Larsen (1977) pointed out that the criteria Brainerd used to assess performance is biased toward detecting developmental sequences vis-a-vis developmental synchronies. More critical, however, is the selection of tasks used to measure ordinal, natural, and cardinal number. These concepts are complex, and attempts to measure children's understanding of them should therefore include a comprehensive array of tasks designed to tap each of their various components. Brainerd, however, employs a narrow set of tasks which do not reflect the full meanings of these concepts. The ordinal number problems usually involve some form of a transitivity task; the natural number problems are usually a series of arithmetic number facts; and the cardinal number problems generally involve a conservation-type task. The central question is whether the observed ordinal-natural-cardinal sequence is a function of basic competence or simply reflects differences in difficulty of the selected tasks due to nonessential task variables. Brainerd (1976, 1979) cited the results of a study by Gonchar (1975) as supporting his position. Although Gonchar found the same developmental sequence for Brainerd's tasks, a close synchrony was found when other ordinal tasks were used. Gonchar concluded that Brainerd's

ordinal-cardinal sequence is primarily a performance distinction between the tasks used to measure each concept.

QUANTIFICATION SKILL THEORIES

The theoretical positions of Piaget and Brainerd, while differing in some important respects, agree that the development of certain logical reasoning abilities are necessary for the acquisition of a complete number concept. Children construct number through the application of basic mental operations, rather than acquiring number through the application of quantification skills. It is in this respect that these positions differ most significantly from those to be reviewed in this section.

Ginsburg (1975, 1976, 1977a, 1977b), like Piaget and Brainerd, believed that the number concept is not complete without certain reasoning abilities including conservation. However, Ginsburg's work is directed toward describing what young children <u>can</u> do, rather than characterizing young children strictly in terms of their intellectual deficiencies.

Ginsburg suggested that the development of preschool children's knowledge of number concepts can be portrayed as a progression through two cognitive systems. System 1 is <u>informal</u> in that it develops outside of formal school instruction; and it is <u>natural</u> since it does not depend on social transmission or specific cultural experiences. Children who are operating within this cognitive system are able to discriminate between numerosities in terms of "more" and "less," based on well-developed perceptual skills. Since these judgments are based on perceptual cues, changes in things such as length or density of the displayed sets which are inconsistent with actual numerosity may lead to erroneous responses. Consequently, System 1 does not yield a mature number concept.

System 2, like System 1, is informal, i.e., it develops prior to formal instruction. However, System 2 is not a natural system since it depends upon socially transmitted knowledge. Counting is the primary characteristic of System 2 and provides the child with a widely applicable and reliable quantification skill. Ginsburg (1977b) believes that counting plays such an important role in children's concept of number that even after formal instruction, "the great majority of young children interpret arithmetic as counting" (p. 13).

At first children learn a portion of the number sequence by rote and then begin searching for rules which will generate the entire sequence. Eventually, counting becomes a rule-governed activity, but its consistent and accurate application depends upon the development of logical reasoning abilities which are necessary to make one-to-one correspondences between the counting numbers and the objects, and to plan strategies for enumerating each object once and only once.

Without these logical abilities children are believed to commit certain predictable errors. First, number is treated as a name rather than an aribtrary and temporarily assigned label. This leads to the erroneous assumption that the order in which the objects are counted makes a difference. Counting is also tied to concrete contexts and is applied only to collections perceived to be homogeneous. Finally, the dependence on perceptual cues reappears when the cues are salient; and reliance on the more accurate counting skill is abandoned.

Counting skills first are applied successfully to small collections and then to larger ones. They can be used early on to discriminate between two static sets (i.e., to judge more, less, or equal) even if the

perceptual cues are misleading. However, prior to the development of the required logical reasoning abilities, counting is not helpful in solving Piaget's conservation task. Consequently, the counting skill itself is believed to be insufficient for acquiring a complete number concept. "A mature concept of number requires more than just counting or the appropriate number language: it requires mature thought" (Ginsburg, 1975, p. 136).

Whereas Ginsburg acknowledges that conservation and other basic logical reasoning abilities play a significant role in the development of a complete number concept, other theoretical approaches propose that children's concept of number grows strictly out of the acquisition and application of certain quantification skills. Number concepts do not depend on the development of more basic logical reasoning abilities, in fact it is the other way around. Gelman (1972b, 1977, 1978; Gelman & Gallistel, 1978) presented one of the more carefully reasoned and documented statements of this position. She began by emphasizing the distinction (also made in this paper) between processes of quantification and processes of reasoning. She distinguished further between reasoning about specified numerosities (collections that have been quantified) and unspecified numerosities. To Gelman, the definition of a number concept does not include operating with unspecified numerosities. Consequently, a discussion of number concept development can be carried out entirely within the context of numerosities that can be accurately represented. In fact, Gelman suggests that to do otherwise would shortchange the child's proficiency in operating with number.

Confining the focus of investigation to numerosities which the young child can quantify leads to an in-depth analysis of quantification skills which yield accurate representations. Gelman's research (Bullock & Gelman, 1977; Gelman, 1972a; Gelman & Tucker, 1975) reflected this emphasis. Counting is believed to be the basic and primary quantification skill. It serves to reliably determine the numerosity of sets and thereby defines the domain within which children first learn to operate with number.

The development of the counting skill over the preschool years is guided by the presence of five counting principles which define a successful counting procedure. The first of these is a one-to-one correspondence principle which requires that each item be assigned one and only one label. This involves partitioning the objects at each count into those which have been counted and those yet to be counted, tagging each item with a unique label, and synchronizing these two activities. The second counting principle is the stable-order rule. This points out the need to use the same number list for every new count. The list need not be the conventional number list; it need only contain the same tags, assigned in the same order each time a collection is counted. The third principle is the cardinal principle. This says that the final tag applied to a collection identifies the numerosity of that collection.

The first three principles describe the counting mechanism; the fourth generalizes these "how-to-count" principles to any collection of physical and nonphysical entities. This abstraction or "what-to-count" principle concerns the range of applicability of the first three. The

fifth and final principle is one of order-irrelevance. This principle states that the order in which the items are tagged has no effect on the counting process. It includes the idea that a counted item does not permanently retain the number name which it was assigned and that the same cardinal number results regardless of the order of enumeration.

These five counting principles are believed to form a scheme, in the Piagetian sense, which motivates and guides the child's developing counting behavior. A unique aspect of Gelman's theory is the conjecture that these principles are "wired in" and unfold with development, much as Chomsky's (1965) language principles (Gelman & Gallistel, 1978). Thus, the principles are believed to precede acquisition of the related skill so that children's behavior is rule-governed rather than capricious. In other words, young children possess counting principles in search of appropriate skills. Gelman's conjecture about the origins of these principles also means that development is primarily a matter of perfecting skills rather than acquiring new principles. More efficient and accurate execution of counting skill is in fact seen to be the major target of preschool children's number concept development.

The five counting principles emerge in an identifiable sequence. Children first show evidence of the stable-order principle, followed by the one-to-one correspondence principle and the cardinal principle. The abstraction principle is also presumed to become functional at about this time. The development of these how-to-count and what-to-count principles overlap to a significant degree, but it is only after they are well-established that the order-irrelevance principle appears. As Gelman noted, this principle involves a good deal of reasoning as well as skill

execution. Consistent with the position that quantification skills precede reasoning about number, Gelman and Gallistel (1978) suggested that "being a reasonably good counter is a necessary but not a sufficient condition for getting a high score on the 'doesn't matter' [order-irrelevance] test" (p. 148).

Several behaviors accompany the development of the counting procedure. First, there seems to be a move from overt to covert action as the counting routine becomes more efficient and reliable. Initially, children point when counting, possibly to help coordinate the one-to-one principle or to keep attention focused on the task. Counting aloud is also a popular technique which may serve the same function. As counting becomes routinized and requires less attention, children dispense with assigning all but the last tag (the cardinal number) aloud.

This description is strikingly similar to Davydov's (1975) characterization of the counting process. Initially, counting depends upon the presence of objects and exaggerated hand movements. With practice the hand movements become abbreviated and are eventually replaced by counting aloud, often accompanied by slight head movements. These overt actions finally disappear as counting becomes internalized. Thus, the development of the number concept involves the gradual internalization of the counting process.

A second important characteristic of the development of counting noted by Gelman is its gradual extension from very small to large numbers. Children can reliably count small sets before they can apply this skill to larger sets. This fact becomes particularly significant

when considering the development of reasoning abilities. Since Gelman maintains that children can only reason with what they can quantify, it follows that children's reasoning abilities are first operational with small numbers and only gradually extend to larger numbers.

> The child's arithmetic reasoning is intimately related to the representations of numerosity that are obtained by counting. The domain of numerosities about which the child reasons arithmetically seems to expand as the child becomes able to count larger and larger numerosities. (Gelman & Gallistel, 1978, p. 72)

Gelman identified three reasoning principles which govern the way children think about number once it has been abstracted from the set. The first concerns relations and the ability to recognize equivalent and nonequivalent numerosities. Equivalence can be established in two ways: by quantifying each set separately and comparing their cardinal numbers, or by setting up a one-to-one correspondence between elements of the sets. Although the latter procedure is the one used in formal mathematics, the former is presumably preferred by young children. Gelman suggests that this difference is the greatest disparity between children's arithmetic and formal arithmetic. It is proposed as a striking counterexample to the hypothesis that the development of children's cognitive structures mirror the logical structures of the discipline. It also assumes that young children can appreciate numerical equivalence before they can establish or conserve one-to-one correspondences. Gelman suggested that Piaget's number conservation task requires more than reasoning about numerical equivalence. It requires reasoning about the equivalence between unspecified numerosities. The first kind of reasoning Gelman called arithmetic reasoning and the second kind Gelman called

algebraic reasoning. Arithmetic reasoning functions within the domain of quantified sets but not outside of it. Gelman maintained that children perceive the conservation question to be a question about correspondences and unspecified numerosities and consequently, do not apply their arithmetic reasoning principles to solve the task.

While the first arithmetic reasoning principle concerned relations between numerosities, the second deals with operations on numerosities. Gelman believes that young children can distinguish between transformations that are relevant and irrelevant to quantity as long as they can determine the numerosity of the set in question. This principle provides the basis for children's understanding of arithmetic operations such as addition and subtraction and depends only on a reliable counting procedure.

The third arithmetic reasoning principle is closely related to the second and says that children not only recognize relevant transformations but that they also can specify an inverse transformation which will "undo" the effect of the first. If a certain number of objects have been added to a set, the reversibility principle says that this effect can be nullified by removing the same number of elements from the set. Again, the existence of this principle depends upon a counting procedure which will reliably determine the numerosity of the set.

In summary, Gelman's approach to the study of early number concepts was an attempt to uncover and carefully describe young children's proficiency with number. She made an important distinction between reasoning about numerosities and reasoning about relations or unspecified

numerosities. The first depends upon, and follows closely behind, the development of counting skills. Therefore, young children were pictured as logical reasoning arithmeticians who deal with small numbers.

As children's principle-governed counting behavior increases in efficiency it is extended to larger and larger sets, and with this extension comes the ability to reason with larger and larger numbers. Reasoning about unspecified numerosities is not a simple extension of this process but involves a qualitatively new form of thought. This form of reasoning, suggested Gelman, is not required for the development of early number concepts.

A second line of research which assumes number to be the outgrowth of quantification skills can be characterized as a hierarchic skill integration approach. Adherents of this approach focus on the acquisition of separate skills such as subitizing (immediate perceptual apprehension of number), counting, one-to-one correspondence, and estimating. Their concern is with the order in which these skills are acquired and the role each of them plays in the development of the others. While many researchers have attempted to document the acquisition of these skills (Dixon, 1977; D'Mello & Willemsen, 1969; Riess, 1943b; Schaeffer, Eggleston, & Scott, 1974; Siegel, 1971; Wang, Resnick, & Boozer, 1971; Wohlwill, 1960b; Young & McPherson, 1976), the work of Klahr and Wallace (1976) represents the most concerted effort to develop a theoretical rationale for this approach.

Klahr and Wallace (1976) postulated three distinct quantification processes--subitizing, counting, and estimating. The function of these

processes is to generate quantity symbols for mental manipulation, which represent the numerosity of a given set. These processes or skills are hypothesized to develop in an invariant sequence. Subitizing is the first skill to be acquired and comprises the basis for children's understanding of number. Subitizing also plays a vital role in the later development of counting and estimating. These latter two skills develop concurrently, but since estimating requires the acquisition of several additional component skills, it reaches maturity later than counting.

The developmental goal of the quantification processes is to provide consistent output, i.e., to reliably determine the numerosity of sets and generate the same symbol for equivalent sets. The mechanism whereby this consistency is thought to be achieved places this approach in distinct contrast to the positions reviewed previously. Klahr and Wallace (1976) believe that the quantification skills become reliable through the detection of regularities or consistencies in the environment. This means that children <u>discover</u> number by abstracting it from empirical activities. Using McLellan and Dewey's (1896) terminology, number is taken out of objects rather than put into them. The detection of regularities is also presumed to explain the way in which earlier developing skills facilitate the acquisition of later skills. Counting begins to achieve reliable output as it is applied to small sets within the child's subitizing range and both processes are observed to yield the same result. Likewise, estimating develops through detecting the consistencies of double-processing judgments in overlapping domains.

Klahr and Wallace's emphasis on the discovery of number through empirical abstraction contrasts sharply with the positions taken by Piaget and Gelman. Piaget (1952) maintained that children construct number through the deployment of mental operations. Therefore, for Piaget, the development of number concepts does not depend on specific learning experiences, but rather on the development of logical operations and the reorganization of mental structures. Gelman (Gelman & Gallistel, 1978) dealt with this problem in yet another way. While she maintained that quantification skills develop prior to the logical abilities identified by Piaget, she differed with Klahr and Wallace (1976) on the mechanisms which motivate their development. Rather than depending on the detection of environmental regularities, Gelman postulated the a priori existence of several counting principles which govern the acquisition process. Counting is considered the essential quantification skill, and development consists of refining the given logical principles. Gelman believes that while specific experiences are necessary to bring the counting procedure within culturally accepted norms (e.g., using the conventional verbal sequence "one, two, . . ."), they are not required to demonstrate its basic rules of usage.

A corollary of Klahr and Wallace's (1976) approach is that the quantification skills develop gradually and are first operational with small numbers. The extension from small to large numbers is reflected both in the developmental sequence of individual skills (i.e., subitizing-counting-estimating) and in the increase in proficiency within a particular skill. The idea that counting is first applied correctly to small numbers is

consistent with Gelman's (Gelman & Gallistel, 1978) conclusion. One reason for this sequence of development may be that children have difficulty systematically partitioning the objects into "already counted" and "to be counted" sets as they are counting (Potter & Levy, 1968; Wang, Resnick, & Boozer, 1971). These component skills develop as other skills are practiced and eventually automated, reducing their demand on the child's attention and working memory, and freeing memory space to concentrate on other skills such as the "partitioning" skill (Schaeffer, Eggleston, & Scott, 1974).

Along with the acquisition of skills for quantifying individual sets comes the ability to make comparisons between sets in terms of relative numerosities. Klahr and Wallace (1976) made an important distinction between two comparison processes: one process compares internal representations or "symbols" of the sets which are generated by the quantification skills, while the other compares the actual entities of the sets. Since the quantification skills of subitizing and counting are the earliest emerging processes with which the child deals with number, and since these produce internal representations of sets, the first comparison processes are those which operate on symbols. Therefore, the first method a child has to compare the numerosities of two sets is to quantify each set separately and then compare the two numerical representations of these sets. Dealing with one-to-one correspondences is not yet possible since it requires a process which compares the external collections or actual entities of the sets without mediating symbols. This comparison process is believed to be acquired after the quantification

skills and the symbol comparison processes are functional. This hypothesized developmental sequence is used to explain why children who have well-developed counting skills still fail Piaget's number conservation task. The conventional task format elicits the child's immature correspondence comparison process or the estimating quantification skill which is not yet reliable. In either case, the misleading perceptual cues lead to an erroneous response.

In summary, Gelman and Gallistel (1978) and Klahr and Wallace (1976) suggested that quantification skills are acquired and become proficient well before the logical reasoning abilities identified by Piaget are operational. However, they suggested more than that. Not only do these skills happen to develop earlier than Piaget's reasoning abilities, they necessarily develop earlier. According to Gelman, children can reason only with what they can quantify. Therefore, the development of reasoning abilities depends upon the prior acquisition of quantification skills. This is the fundamental difference between the views of Gelman and Klahr and Wallace, and those of Piaget, Ginsburg, and Brainerd.

RECURRING ISSUES

Several important issues surrounding the development of early number concepts have attracted the continuing interest of investigators. Some of these were alluded to in the previous discussion, but three issues in particular deserve further consideration. One is the question of the initial process by which children apprehend number, i.e., the debate on the developmental primacy of subitizing versus counting. A second issue focuses on the conservation phenomena--how does this ability come

about and why is it important? The third issue bridges the gap between this section of the paper and the next. It concerns the role played by the early emerging number skills and concepts in the acquisition of arithmetic operations and other school-related mathematical skills.

Counting or Subitizing: Which Comes First?

History has recorded a continuing debate on the developmental primacy of counting versus subitizing. The purpose of this discussion is not to provide an exhaustive review of this debate since comprehensive reviews already exist (Brownell, 1941; Gelman & Gallistel, 1978; Ginsburg, 1975; Klahr & Wallace, 1976; Martin, 1951). The aim is rather to highlight some of the important considerations which have led to the adoption of one view or the other. As mentioned previously, subitizing is generally defined as the immediate visual apprehension of number. The task used to measure a person's subitizing ability often consists of a series of cards with a different number of dots randomly positioned on each card. The person is shown the cards, one at a time, and is asked to determine the number of dots on each card without counting. The criterion is either the time it takes the person to respond (measured in milliseconds) or the error rate given a constant stimulus exposure time (e.g., two seconds per card).

Graphing the response times for adults on this type of task yields noticeably different functions for small numbers than for large ones. The slope of the graph for the set of numbers up to 6 ± 1 is relatively shallow while that for numbers larger than six is much steeper. This discontinuity at 6 ± 1 appears whether the criterion is response latency,

confidence of judgment, or error rate (Klahr & Wallace, 1976), a result often interpreted as evidence that subitizing is used for small numbers (less than six or seven) and counting is used for larger numbers. Klahr and Wallace hypothesized further that since young children can quantify small sets before they quantify large sets, with the critical value between five and seven, it is likely that children initially quantify sets by subitizing.

Gelman and Gallistel (1978) interpreted this evidence differently, and cited additional evidence to support their position that children count before subitizing. They questioned why, if children subitize small numbers, the slope of the graph is not zero, and furthermore, why the differences in response times follow an orderly progression with an increase in set size. They suggested that the data are explained instead by postulating a rapid counting procedure which is perfected early on with the amount of practice most children experience. This explanation is more consistent, they said, with the ubiquitous tendency of young children to count. Evidence for the prevalence of young children's counting has been gathered by Gelman and associates using a "magic" task (Bullock & Gelman, 1977; Gelman, 1972a; Gelman & Tucker, 1975). Children as young as two years were found to spontaneously use a counting procedure to quantify small sets.

For Gelman and Gallistel, subitizing is not viewed as a low level or primitive process, but rather as a sophisticated procedure for grouping objects visually, thereby increasing the efficiency of quantification. It is a later acquisition, used by older children along with

counting, to quantify visually perceived sets of objects. This idea is similar to that proposed by Brownell (1928). In Brownell's study, children in grades one, two, and three were shown the conventional stimulus cards for five seconds each. Results showed a gradual increase in error rate with the younger children as the number of dots increased, but a sporadic error pattern for the older children. Brownell inferred from these two error pattern types that younger children were counting to quantify the sets while the older children were using perceptual grouping, addition facts, and counting to determine the cardinalities. Follow-up interviews with individual children were reported to confirm these hypotheses. Additional evidence was also obtained by readministering the stimuli with a shorter (three second) exposure time. Error rates of the younger children increased while those of the older children remained the same. Again, Brownell interpreted this as evidence for the increased tendency with age to use subitizing as a sophisticated grouping procedure in the service of more efficient quantification.

The inconclusive nature of the results and the continuing debate about the sequence in which these skills develop stem from two methodological problems. One is the difficulty of inferring process used from product data. Most of the evidence which bears on this question consists of average response latencies or error rates. The process which was used to produce these data must be inferred; no direct evidence is available on the quantification process itself. The problem is that usually there are several processes which may have produced a given set of results and is accentuated by the fact that the processes in question

are executed so quickly that individuals have a difficult time monitoring them and giving retrospective accounts of their use. Consequently, the investigator gets little help from asking subjects to explain how they quantified a particular set.

A second methodological problem is one of obtaining information from the population of interest. Since the question concerns the earliest form of quantification, the population of interest is children less than two years of age. It is difficult to gather reliable data from children this young, so the usual approach has been to infer certain performance characteristics of young children from the results of older subjects. Coupled with the first methodological problem, this added inference step makes the evidence equivocal at best.

Before moving to the second major issue, mention should be made of another perceptual process which is distinguished from subitizing. This process involves the comparison of two visually displayed sets rather than the quantification of a single set. The task requires the subject to determine which of the two sets has more or less and, therefore, is a question about relative numerosity rather than absolute numerosity. Young children are apparently quite proficient in perceptually determining the larger or smaller of two sets displayed as randomly positioned dots on two different cards. Ginsburg (1975) cited evidence that children as young as four years can distinguish the larger of two sets which differ by only one or two with set size as large as 15. Estes and Combs (1966) reported similar data. However, it may be that children are not attending to number per se in these tasks, but rather to other correlated cues

such as area or brightness (see Trabasso & Bower, 1968). Therefore, while this perceptual ability may be unrelated to number, it may be an early developmental form of children's ability to deal with equality and inequality relationships between sets.

Conservation of Number: How Does it Develop?

"The failure of children younger than 5 to conserve . . . is one of the most reliable experimental findings in the entire literature on cognitive development" (Gelman & Gallistel, 1978, p. 1). When and how does this much discussed and frequently studied phenomenon occur?

Generally, children begin performing successfully on number conservation tasks by age five or six. A much debated study by Mehler and Bever (1967) claimed to show evidence of conservation much earlier than this, at about two years of age. Although their general findings have been replicated (Bever, Mehler, & Epstein, 1968; Calhoun, 1971), several investigators (Beilin, 1968; Piaget, 1968; Rothenberg & Courtney, 1968; Willoughby & Trachy, 1972) have conducted related studies and concluded that Mehler and Bever were not dealing with true conservation and that their findings were artifacts of the nonconventional task they used to assess conservation.

Whatever the resolution of this debate, it is true that certain task variables have significant effects on children's number conservation performance. Task characteristics such as number of objects employed (Gelman & Gallistel, 1978; Zimilies, 1966), interest of stimuli (Roberge & Clark, 1976), nature of the relationship between corresponding objects (Piaget, 1952), salience of misleading perceptual cues such as length

and density (Brainerd, 1977; Miller & Heller, 1976), and experimenter expectancy (Hunt, 1975) all have been shown to affect task performance, although contradictory evidence exists in most cases. The one conclusion which can be drawn from this bewildering collection of results is that children cannot be labeled simply as conservers or nonconservers. Young children often conserve number under some conditions but not under others. Therefore, it is difficult to establish an age at which children first "conserve."

However, there is a point before which a child will fail the number conservation task regardless of context. How is it that every child moves from this universal nonconservation ability to increasingly successful conservation performance in a wide variety of task situations? The popularity of this question as a topic of study is probably due to the importance which Piaget ascribed to this phenomena. For Piaget (1952), number conservation is the hallmark of children's acquisition of the number concept. It develops as children begin to decenter their attention and move from focusing on only one dimension to coordinating several dimensions of the stimulus situation. In the number conservation task, this means recognizing that the decrease in density compensates for the increase in length when one row of objects is spread apart. The tendency of young children to center on only one dimension, usually length, in the number conservation task is a well-documented fact (Baron, Lawson, & Siegel, 1975; Brainerd, 1977; Lawson, Baron, & Siegel, 1974) and has even been observed by recording children's eye movements during task solution (O'Bryan & Boersma, 1971).

At a more abstract level, Piaget described the development of conservation as the modification or reorganization of mental structures. Children at this stage of development acquire the ability to reverse their thought processes, i.e., to mentally reverse the transformation they have just witnessed and return the elements to their original state. This ability, to run back and forth in one's mind, characterizes the onset of operational thought. Conservation is simply a reflection of this new mental structure.

Piaget's explanation for the development of number conservation suffers from a certain abstractness and high level of inference. It is difficult to validate the presence of "mental structures." "The absence of a precise process-performance link contributes to the extreme difficulty of putting Piaget's account of transition to an experimental test" (Klahr & Wallace, 1976, p. 4). Partially because of this problem, several alternate explanations have been advanced for the development of conservation. Klahr and Wallace described the acquisition of number conservation in information-processing terms. Like their explanations of the growth of quantification skills, they suggested that children learn how to conserve by detecting regularities in their experience with quantity. As children perfect their quantification skills, they are able to determine the numerosities of sets before and after transformations. In this way, they learn to identify those transformations which are irrelevant to number, i.e., they learn to conserve. Since subitizing is believed to be the first available quantification skill, and since only small sets can be quantified using this skill, conservation is first

functional with small numbers. Once children achieve conservation with small numbers, they are able to develop more rapidly the additional quantification skills (counting and estimating), since the acquisition of these skills also depends upon detecting empirical regularities. Therefore, conservation is seen to facilitate the development of counting rather than vice versa.

The problem which accompanies Klahr and Wallace's description of conservation acquisition is that it is difficult to see how the detection of regularities can lead to conservation when conservation is the first sign that regularities are detected. That is, how can children observe a "no change" condition in the stimulus after the transformation if they do not yet conserve, since by definition, conservation represents precisely this ability. (See Wallach, 1969, for a further discussion of this problem.) This is, in effect, a type of circular reasoning. The development of conservation is being explained by its own definition.

Although Gelman and Gallistel (1978) did not offer an explanation of how conservation is achieved, their description of why children fail in conservation is sufficiently important to warrant some discussion. The thesis of Gelman's work is that young children can deal proficiently with small numbers, i.e., with sets of objects they can quantify. This proficiency includes the ability to distinguish between number relevant and number irrelevant transformations. The question is why children do not apply these reasoning principles to the larger collections of objects traditionally used in the conservation task. Gelman and Gallistel maintained that young children do not evidence their already developed reason-

ing ability in the conservation task because they interpret the question to be one about equivalence relations between unspecified numerosities rather than about the equivalence of two specific cardinal numbers. The authors believe that the perceptual salience of the initial one-to-one correspondence leads children to this interpretation.

The problem for Gelman and Gallistel, who hang so much of children's number concept development on their ability to count, is why children who count in the conservation task, either spontaneously or upon request, still fail to conserve. It seems as though such counting behavior would help the child reinterpret the question as one involving the present numerosities. However, many young children count the two sets correctly and then give a nonconservation response (Carpenter, 1971; Ginsburg, 1975; Wohlwill & Lowe, 1962). Piaget (1952) observed this phenomenon and concluded that there is little relationship between the two: "There is no connection between the acquired ability to count and the actual operations of which the child is capable" (p. 61). Gelman and Gallistel would, of course, disagree with this claim. But it still remains to explain how young children can possess the counting and reasoning skills they attribute to them and fail to conserve number.

This problem is indicative of a more general issue which emerges when discussing the development of number conservation--what is the relationship of conservation to other number skills and concepts. For those, like Gelman and Gallistel, who claimed that children have some substantive number skills prior to conservation, the problem is to explain why these proficiencies do not suffice for the conservation task. For

others, like Piaget, who insisted that conservation marks the beginning of a mature number concept, the problem is to explain how children can become proficient in number skills and concepts before conserving.

Early Number Operations: How Do They Affect Later Mathematics Performance?

Of the early number concepts and skills considered to this point, two have been suggested by educators and psychologists as playing particularly important roles in the acquisition of more complex arithmetic operations. These are conservation and counting. This section reviews the research on the relation between children's conservation ability and their ability to add and subtract. A more complete review of the relationship between performance on Piagetian tasks and arithmetical performance can be found in Hiebert and Carpenter (in press).

Many investigators have taken a global approach in studying the relationship between conservation and mathematical performance. A frequent technique is to administer a battery of Piagetian tasks and a school mathematics achievement test, either concurrently or several months or years apart, and correlate the scores of these two measures. These data usually show high, positive correlations between conservation responses and achievement scores (Dimitrovsky & Almy, 1975; Dodwell, 1961; Kaminsky, 1971; Kaufman & Kaufman, 1972; Nelson, 1970; Rohr, 1973; Smith, 1974), although some low correlations have been reported (Cathcart, 1974; DeVries, 1974; Pennington, 1977).

Some studies have considered the relationship between number conservation and children's facility with specific mathematical skills or concepts.

Steffe (1970) and LeBlanc (1971) observed first-grade children's addition and subtraction skills, respectively, and their relationship to number conservation ability. Both found that conservation performance was a significant predictor of arithmetic skill, with low conservation scores associated with especially poor arithmetic scores. Woodward (1978) considered several different types of addition and subtraction problems and found that number conservation was significantly related to first-grade children's performance on all types of problems except subtraction problems involving missing differences. Johns (1974), on the other hand, found only a few significant correlations between number conservation and the subtraction skills of first, second, and third graders; and Michaels (1977) reported that some specific addition and subtraction abilities emerge before the ability to conserve number. The fact that relationships seem to exist between number conservation and only certain types of arithmetic problems suggests that different skills or concepts may make different demands on conservation ability.

A study by Hiebert, Carpenter, and Moser (1982) provides additional information by analyzing the processes that children at different levels of conservation ability use to solve basic addition and subtraction problems. On most problems, nonconservers used an appropriate strategy less often than conservers, but every strategy identified was used by at least some nonconservers. As a consequence, Hiebert et al. concluded that conservation is not a prerequisite for solving basic addition and subtraction problems or for acquiring advanced solution strategies.

The evidence reviewed to this point is only suggestive since the positive correlations provide little insight into the reason for the relationships. They do not indicate that number conservation is a prerequisite for learning arithmetic skills or concepts. The high correlations may result from the fact that both require the same underlying abilities or that they are both highly correlated with general mathematical ability. The question of interest is still whether, in fact, number conservation is required to learn certain mathematical skills. Lovell (1966) suggested a theoretical basis for believing that conservation may be required to understand arithmetic operations. Since success on the conservation task implies an understanding of those transformations which are irrelevant for number, it also indicates a basic knowledge of relevant transformations (e.g., addition and subtraction). However, experimental studies involving instructional components are needed to determine whether conservation is an essential prerequisite to such understanding.

One such study, conducted by Mpiangu and Gentile (1975), investigated the effect of number conservation on kindergarten children's ability to learn certain arithmetic skills. Problems on the arithmetic pre- and posttest involved numbers 0-10, and most of them required rote- or point-counting skills. (Rote counting consists of recitation of the counting numbers in correct sequence; point counting involves establishing a one-to-one correspondence between the counting numbers and a set of markers, and labeling the set with the appropriate cardinal number.) Although number nonconservers performed lower than conservers on the arithmetic pretest, and still performed lower on the posttest given after 10 arith-

metic training sessions, they showed a similar amount of gain from pre- to posttest. The authors interpreted this as evidence that conservation (a) does not affect children's ability to benefit from arithmetic instruction, and (b) is not a necessary condition for mathematical understanding.

Steffe, Spikes, and Hirstein (1976) contested this conclusion after conducting a prolonged instructional study with first-grade children. Following about 40 hours of arithmetic instruction over a three-month period, all children were tested on 29 individual measures which were clustered into seven achievement variables. Six of these variables assessed numerical skills such as working with cardinal and ordinal numbers; solving orally presented addition and subtraction problems with and without objects; and counting at the rote, point, and rational levels (rational counting is evidenced by counting on or counting back to solve a numerical problem). The results of the study are complex and difficult to summarize. However, several of the major findings follow: (a) number conservers performed significantly better than number nonconservers on those tasks which required rational counting; (b) number conservation was not required to perform tasks solvable by rote counting; and (c) with special training, number conservation was not required to master tasks solvable by point counting.

The authors concluded that children who differed in conservation ability differed in the benefit they derived from instruction. The learning experienced by the number conservers was qualitatively different than that of the nonconservers. Number conservers were able to acquire

rational-counting skills and apply them to solve a variety of problems. Nonconservers, on the other hand, demonstrated task-specific learning and used rote- and point-counting procedures. The authors suggested that the conclusions of Mpiangu and Gentile (1975) suffer from overgeneralization. While conservation may not affect the learning of simple skills, it is important for learning more advanced and logically based concepts and skills.

One feature of the study by Steffe et al. (1976) which makes the results difficult to compare with other studies is the use of a nonconventional conservation task. The task required children to discriminate between two sets of objects presented in a visual display. No transformation was performed on either of the sets. As mentioned earlier, this perceptual discrimination is a relatively easy task for many children as young as 4 years old (Beilin, 1968; Siegel, 1971; Wohlwill, 1960a). Ginsburg (1975) referred to this as a simple equivalence task, in contrast to the traditional conservation, or complex equivalence task. Although the relationship between the two is not entirely clear, Ginsburg argued that success on the former does not imply a full understanding of number.

In general, it appears that performance on conservation tasks is correlated with performance on arithmetic tests. Conservers are generally more successful in solving a variety of addition and subtraction problems and use advanced strategies more often than nonconservers. There is little evidence, however, that the ability to conserve is required to learn any of the basic arithmetic concepts or skills that are presently included in the school mathematics curriculum.

BASIC ADDITION AND SUBTRACTION CONCEPTS

Research on addition and subtraction has focused primarily on an analysis of the quantification skills involved in adding and subtracting. However, some of the research on the logical reasoning abilities required to acquire a complete understanding of number has also included an analysis of basic concepts underlying addition and subtraction. This research has been revised in some detail by Starkey and Gelman (1982). Therefore, this research is summarized only briefly here.

Perhaps the most basic principle underlying addition and subtraction is that joining elements to a set increases its numerosity and removing elements decreases its numerosity. Brush (1978) and Smedslund (1966) and others have found that most kindergarten and older preschool children understand the effect of these transformations, and some studies have found that children as young as three years could successfully compare sets that had elements joined or removed (Cooper, Starkey, Blevins, Goth, & Leithner, 1978). In fact, the earliest age at which children understand the effect of these transformations has not been identified (Starkey & Gelman, 1982).

Piaget (1952) has argued that an operational understanding of addition also requires that a child recognizes that a whole remains constant irrespective of the composition of its parts. He found a stagewise development of this concept that paralleled the development of conservation. In the initial stage, children did not realize that in a set of eight objects divided into two subsets of four objects, each was equivalent to a set of eight objects divided into sets containing one and seven

objects. In this stage, children responded primarily on the basis of misleading spatial cues. In the second stage, children could often solve the task correctly, but only after empirical verification. It was not until the third stage at about age seven that children logically recognized that the composition of the set did not affect the number in the set (i.e., 4 + 4 = 1 + 7). Several other experiments, concerned with partitioning sets into equally numerous arrays, found similar patterns of development.

It has also been proposed that a complete understanding of addition and subtraction requires that children understand the basic properties of each operation. Two properties that have been the subject of several studies are inversion and compensation (Brush, 1978; Cooper et al., 1978; Smedslund, 1966). An understanding of inversion means that a child recognizes that the effect of adding elements to a set can be offset by removing the same number of elements. Understanding compensation requires that the child recognize that adding elements to one of two equivalent sets can be compensated for by adding the same number of elements to the other set.

To test understanding of these properties, children were first to establish the relation between two initial arrays. In some cases, the arrays contained the same number of elements and in others, one array contained more elements than the other. In general, a child would not determine the exact number of elements in the arrays, but rather the relation between them. Often the arrays contained too many elements to

make counting easy or they were partially covered to prevent counting. This was done to test children's understanding of the properties of inversion and compensation and not their ability to calculate the exact number of elements following a transformation. Once the initial relationship was established, elements were added to one of the arrays. Then, if it was an inversion task, elements were removed from the same set. If it was a compensation task, elements were added to the other set. In some cases, the second transformation involved the same number of elements as the first, in some cases it did not.

The ability to solve inversion and compensation problems develops over a number of years. Young children at about age 3 give a primitive solution that is based exclusively on the last array transformed. If the last transformation involves joining elements to an array, that array is thought to contain the most elements, regardless of the initial relation between the arrays or the previous transformation. Similarly, if the final transformation involves removing elements from one array, that array is thought to contain fewer elements than the other. Before a child develops an operational understanding of inversion and compensation, there is an intermediate stage in which the child relies on qualitative solutions that take into account all the transformations but not the number of elements in the transformation. At this stage a child would believe that adding two elements to a set from which three elements had just been removed would return the set to its initial state.

To summarize, young children have nonperceptual, noncounting procedures for solving certain types of addition and subtraction problems. The basic principles underlying these strategies appear to develop in a

pattern that is similar but not synchronous with the development of conservation and other concrete operational concepts. Additional work is needed to give a more detailed description of these strategies and their relation to the quantitative strategies described in the following section of this review.

With a few exceptions, most notably Piaget (1952), most research on basic logical operations underlying addition and subtraction has been conducted within the past 15 years. By contrast, research that directly assessed children's ability to add and subtract, dates back to the beginning of the century. The earliest research focused primarily on problems presented symbolically, but recently, much of the research has examined children's solutions of simple addition and subtraction word problems. There are clear similarities in how children solve symbolic problems and word problems; but because there are some important distinctions between them, symbolic problems and word problems are discussed separately.

SYMBOLIC PROBLEMS

RELATIVE DIFFICULTY OF BASIC FACTS

Most of the early research on addition and subtraction investigated children's ability to compute and was concerned exclusively with identifying which problems children could and could not solve. The largest group of early studies attempted to rank the relative difficulty of the 100 addition and subtraction number combinations (basic facts). Among the empirical attempts at ranking combinations by difficulty were those by Holloway (1915), Counts (1917), Smith (1921), Clapp (1924), Batson and Combellick (1925), Washburne and Vogel (1928), Knight and Behrens (1928), Thiele (1938), Murray (1939), and Wheeler (1939). This research has been reviewed in some detail (Brownell, 1941; Buckley, 1975; Buswell & Judd, 1925; Grouws, 1972a; Suppes, Jerman, & Brian, 1968). Much of Brownell and Carper's (1943) critique of studies ranking the difficulty of the multiplication combinations is also relevant.

The studies reviewed attempted to determine a ranking of the 100 canonical addition facts and the 100 canonical subtraction facts when presented in abstract form. The motivation for this research was to provide guidelines for organizing instruction. If certain facts were "intrinsically difficult," they might require more practice, their introduction might be delayed, pairs of facts might be taught together, etc. However, the studies did not generate consistent rankings and did not provide a coherent empirical basis for the design of instruction.

Brownell (1941) pointed out many conflicting results and inconsistencies in the rankings in these studies. He noted that ranks for 0 + 3

were 1, 8, 14.5, 32, 76, and 97.5 in six of the studies, and that in two of the studies, a change from 93.0% to 93.8% correct caused a rank difference of over 20 places for the combination 0 + 5. He also noted that two combinations ranked nearly equally could differ by 40 points in percent correct. Suppes, Jerman, and Brian (1968) ranked just the "zero combinations" and found that some of the 19 combinations varied in rank from 2 to 17. Grouws (1972a) concluded that a serious fault of any such rankings is that they exaggerate small differences and mask large differences.

The only clear consistency in the rankings of different studies is that the difficulty of addition and subtraction combinations increases as the numbers get larger. Although there were relatively high correlations between the rankings of several studies (Murray, 1939), there were many conflicting results and inconsistencies (Brownell, 1941).

In addition to attempting to generate a linear ranking of the number facts, some studies attempted to identify factors that account for the relative difficulty of different number combinations. Knight and Behrens (1928) hypothesized that addition facts in which the larger addend is given first (e.g., 6 + 3) would be easier than the corresponding pair in which smaller addend appears first (3 + 6). Their results and the results of other studies failed to confirm the hypothesis. Browne (1906) and Pottle (1937) contended that problems in which both addends are even were easier than problems in which both addends are odd which in turn were easier than problems involving one even and one odd addend. Other studies, however, did not support this hypothesis either (e.g.,

Murray, 1939). In general, the only consistent pattern that emerges is that doubles (e.g., 8 + 8) are easier than other combinations with comparable size addends.

One factor which contributed to the conflicting results was the variety of experimental conditions used in the studies. Brownell and Carper (1943) pointed out that the studies ranking combinations used subjects ranging from beginners to experts, e.g., Knight and Behrens (1928) tested second graders during initial instruction on the facts, Smith (1921) used subjects in grades 3 through 8, and Batson and Combellick (1925) used college graduates and undergraduates. It is unlikely that difficulty rankings would be consistent when based on studies using subjects of varying ages and tested before, during, and after instruction on the facts.

The method (rote or meaningful) by which subjects were taught, administration details such as presentation order, and success criteria were other factors which contributed to the conflicting results. Among the criteria for success were percent correct, the number of trials needed to learn, latency of the first correct solution, latency of correct solutions on a review test, etc. Brownell (1941) also pointed out that a rank for $\underline{+\ 3}^{5}$ might actually be a rank for "3 + 5" if the subject added upward in vertical addition problems. The above factors make comparisons among studies extremely risky.

Brownell (1941) provided a succinct analysis of research on number combination difficulty.

> It may be assumed that all difficulty rankings are authentic for the conditions under which they were obtained and for the techniques by which they were

> determined. And this is precisely why research to ascertain the comparative difficulty of the combinations has been unprofitable. There is no such thing as "intrinsic" difficulty in the number facts; their difficulty is relative, contingent upon many conditions, chief of which is method of teaching, or stated differently, the number, order, and nature of learning experiences on the part of pupils. (p. 127)

Brownell's attack seems to have dampened enthusiasm for this line of research, and after 1941 there were relatively few studies attempting to rank order basic number facts. Researchers continued to be interested in the relative difficulty of different problems, but the emphasis shifted to structural features of the problems or the conditions under which problems are administered.

STRUCTURAL VARIABLES

A number of studies have attempted to determine factors which affect the success rate on "open sentence" problems. Open addition and subtraction sentences (canonical and noncanonical) are often classified into six types by operation and the position of the placeholder. These are $a + b = \square$, $a - b = \square$, $a + \square = c$, $\square + b = c$, $a - \square = c$, and $\square - b = c$.

Six studies (Beattie & Beichmann, 1972; Goren & Poll, 1973; Grouws, 1972a, 1972b, 1974; Hirstein, 1979; Lindvall & Ibarra, 1980a; Weaver, 1971) have generated somewhat comparable data pertinent to the relative difficulty of some of the six open sentence types above. Commonalities among portions of these studies include (a) subjects from grades 1, 2, and 3, (b) the use of constants from the "basic fact" domain, (c) use of sentences with whole number solutions, and (d) use of the "operation-left" form of open sentences. Selected results from these studies which addressed overlapping questions are summarized in Table 1.

Table 1

The Effects of Operation and Placeholder Position on the Difficulty of Open Addition and Subtraction Sentences

	a + b = □	a - b = □	a + □ = c	□ + b = c	a - □ = c	□ - b = c
Grade 1						
Weaver (1971)	65[a]	55	53	46	50	11
Lindvall & Ibarra (1980a)	—[b]	--	83	86	77	26
Beattie & Deichmann (1972)	92	88	80	74	87	75
Groen & Poll (1973)	—	--	77	77	--	--
Hirstein (1979)	75	53	--	--	--	--
Grade 2						
Weaver (1971)	87	80	79	76	79	27
Beattie & Deichmann (1972)	93	94	95	92	94	77
Hirstein (1979)	95	77	71	73	--	--
Grade 3						
Weaver (1971)	93	88	92	88	90	33
Grouws (1972a)	--	--	94	86	88	45

Note. Data are only included for basic fact items with a solution.

[a]Indicates percentage of correct responses.

[b](--) indicates that item did not appear.

Results which are consistent across these studies are:

(1) Canonical addition and subtraction sentences are generally less difficult than noncanonical sentences. Performance on canonical items ranged from 55% to 94% correct across studies and grade levels.

(2) Canonical subtraction items were more difficult than canonical addition items in six of seven instances in Table 1. Such differential performance has been substantiated in numerous studies (e.g., Brownell, 1941).

(3) The two missing addend sentences are of comparable difficulty, with $a + \square = c$ being slightly easier than $\square + b = c$. Performance ranged from 46% to 95% correct.

(4) The missing subtrahend sentence is comparable in difficulty to the missing addend sentences.

(5) The missing minuend sentence is the most difficult and is distinctly more difficult than the others. It was solved correctly less than 50% of the time in all but Beattie and Deichmann's (1972) workbook format, in which teacher assistance was presumably available. Performance in the other studies ranged from 11% to 45% correct.

(6) Performance on all items improves as subjects' age increases. In each of the three studies using subjects at several grade levels (Hirstein's design was longitudinal; Weaver's and Beattie and Deichmann's were cross-sectional), all items used reflected improved performance from one grade to the next.

Several other studies have investigated the effect of operation and placeholder position on performance with open sentence problems. The work of Suppes and his colleagues (Suppes, 1967; Suppes, Jerman, & Brian, 1968; Suppes & Morningstar, 1972) used response latencies and error probabilities to make predictions about the difficulty of the various open sentence types. Suppes' work is not altogether comparable to that in the six studies cited above and some of his conclusions are at odds with the findings of the other researchers. For instance (a) Suppes' subjects were drawn from a population having extensive experience with computer-assisted instruction on these tasks and who displayed near-ceiling performance, (b) latency of response was used as one of the dependent variables so analyses were performed on mean data averaged over individuals, and (c) magnitude of the constants in the open sentences was not strictly controlled.

Suppes et al. (1968) identified several factors which contributed to the difficulty of open sentence problems, including the magnitude of the largest number in the problem, magnitude of the smallest number, the form of the equation, and of steps required to solve the problem. Of these, they found the number of steps was the best predictor of difficulty and this depended upon how many steps were required to transform the problem into canonical form, the number of operations performed, and the number of digits which had to be held in memory. The number of digits held in memory was found to be the most important and the number of operations performed the least important of these three.

Some of the above factors have a questionable influence on difficulty because of a lack of observational evidence that children actually

attend to such factors. In particular, the transformation of a problem to canonical form is not a behavior which has been documented in studies describing children's solution strategies on addition and subtraction problems. Many counting strategies can be employed without any transformation of the problem's original format.

Suppes et al. concluded that $\square + b = c$ was more difficult than $a + \square = c$. They also contended that $\square + b = c$ is more difficult than $\square - b = c$. Both of these findings contradict those of the six studies mentioned earlier. Grouws (1972a) argued that the differences which Suppes et al. found on the two missing addend problems can be attributed to factors other than placeholder position. He noted that a substantially greater number of $\square + b = c$ problems involved two-digit constants than did problems of the form $a + \square = c$, and also pointed out that different samples were used when judging the relative difficulty of $\square + b = c$ and $\square - b = c$ problems. Consequently it appears that Suppes' findings must be interpreted with caution.

Nesher (1979, 1982) also used reaction time to compare types of open sentence problems. Her findings show difficulty to be ranked in the following order (in increasing order of difficulty):

(1) $a + b = \square$

(2) $a - b = \square$

(3) $a + \square = c$ and $a - \square = c$

(4) $\square + b = c$

(5) $\square - b = c$

Nesher's reaction time data are generally consistent with Weaver's and Hirstein's difficulty data for second graders. In this case, reaction

times and percent of correct responses seem to yield similar results, unlike the discrepancies between reaction time data found by Suppes et al. (1968) and direct observation data on strategy use summarized in Table 1.

Symmetric Forms

Two studies (Lindvall & Ibarra, 1980a; Weaver, 1973) have attempted to determine what effect the symmetric presentation of open sentences, e.g., a + b = ☐ vs. ☐ = a + b, has on performance. In both studies, addition and subtraction items with the operation given on the left were consistently easier than those with the operation given on the right.

OTHER VARIABLES

Horizontal/Vertical Format

Although some studies have analyzed both horizontal and vertical formats, Beattie and Deichmann (1972) provide the only systematic comparison of error rates and error types on abstract horizontal and vertical addition and subtraction items. Within both grades 1 and 2, performance was slightly better for vertical than for horizontal items. In three of the four instances (addition at the second-grade level being the exception), they found more computation errors and fewer process (wrong operation) errors in horizontal than in vertical format.

Several factors make it difficult to generalize from Beattie and Deichmann's results. An imprecise categorization of errors is one, and the workbook context is another. It is not clear that workbook items paralleled instructional emphasis; approximately 80% of first-grade workbook items were horizontal while second-grade items were split almost

equally between horizontal and vertical, but no measure of instructional emphasis other than number of workbook items is given.

Engle and Lerch (1971) compared first graders' performance on horizontal and vertical abstract addition items. Performance was slightly higher on vertical than on horizontal items, but the difference was not statistically significant. Engle and Lerch's findings for first-grade addition closely paralleled Beattie and Deichmann's results. In both studies, performance was over 85%, so there may have been a ceiling effect.

Existence of a Solution

Some empirical evidence has been gathered which indicates that the existence or nonexistence of a whole number solution affects performance on open addition and subtraction sentences. However, this evidence is not unequivocal.

In a multiple choice format, Howlett (1974) found no significant differences in first graders' performance on missing addend problems with and without whole number solutions, but performance was slightly better when no solution existed.

Weaver (1972) also reported a significant difference in performance due to the existence of a solution, across grades 1, 2, and 3, but in this case achievement was higher when a solution existed than when no whole number solution existed. Existence of a solution also interacted with grade level and operation. The solution-no solution performance differential was greater at grade 3 than at grades 1 and 2 and was also greater for addition items than for subtraction items (as defined by the

operation present in the open sentence). In fact, first graders correctly solved a - ☐ = c and c = a - ☐ more often when no solution existed than when a solution existed.

Mitchell (1981) administered items of the form ☐ + b = c, a + ☐ = c, a - b = ☐, and a - ☐ = c to first, second, and third graders. For these items, grade 1 performance was better when a solution existed than when it did not, but this difference was less pronounced in grades 2 and 3, This contradicts Weaver's (1972) finding of a greater performance differential at the higher grade levels.

Overall, the evidence suggests that performance is generally better on open sentence items for which a solution exists than on items for which no solution exists. The effects vary considerably, however, with differing placeholder positions, operations, grade levels, etc. The lack of instructional emphasis on problems with no solution makes it difficult to make conjectures regarding other causes for poorer performance on problems with no solution; poor performance may simply be due to lack of opportunity to learn.

Presence of Aids

Little empirical evidence is available regarding the effects of various aids on the solution of abstract addition and subtraction problems. One reason is that much of the research discussed above was conducted in a setting in which no manipulatives were available to the subjects. Besides, some research has indicated that by the time children reach second grade they no longer use manipulatives frequently. In particular, Houlihan and Ginsburg (1981) reported that second graders used manipulatives in

their solutions to two-digit abstract addition problems only 4% of the time. Second graders did not use manipulatives at all on single-digit addition or on single-digit, double-digit problems.

Studies in which manipulatives have been available have generally found that their presence improves performance (e.g., McLaughlin, 1935). The impact of other aids such as pictures is essentially unknown. Grouws (1972a) reported for third-grade subjects that presentation of an accompanying verbal problem did not aid in the solution of abstract open sentence problems. There is little other relevant research available.

STRATEGIES OF SOLUTION

Research Paradigms

Identifying the processes that children use to solve simple addition and subtraction problems is not an easy task. Internal cognitive processes cannot be observed directly, and the problems are sufficiently simple that children themselves often are not aware of how they solved a given problem. Three basic paradigms have been used to study the processes that children use to solve simple addition and subtraction problems. The most straightforward has involved the use of individual interviews.

Use of individual interviews. An individual interview to assess a child's performance on addition and subtraction problems can take several forms. Opper (1977) described Piaget's clinical method, one diagnostic tool for studying children's reasoning. In a true clinical interview, hypotheses are generated about the processes children use to arrive at their solutions and the subject's responses serve as a basis for subsequent tasks and questions from the interviewer. Opper also described a

modification of Piaget's clinical method which she termed the "partially standardized clinical method" (p. 92). This approach uses standard tasks but allows the interviewer freedom to be flexible in subsequent probing related to the child's response.

Alternatives to the partially standardized individual interview also exist and have been used by researchers to study various aspects of children's thinking. Naturalistic observation, teaching experiments, and the case study method (Easley, 1977; Opper, 1977; Stake, 1978) are three of these. Each, however, has advantages and disadvantages. Individual interviews do not generate responses which are as spontaneous as those which derive from naturalistic observation nor do they provide the depth and breadth of data found in the case study approach. On the other hand, the individual interview procedure minimizes occurrences of irrelevant behavior and provides an opportunity to focus on specific thought processes, while retaining sufficient generalizability to make comparisons between subjects and tasks possible.

Researchers who have used the individual interview procedure with young children have often reported difficulty in eliciting or interpreting the child's verbalizations. Menchinskaya (1969) used thinking aloud and introspection to study problem-solving behaviors of first graders but reported that "verbal description of their actions was difficult even for the stronger pupils" (p. 25). Shchedrovitskii and Yakobson (1975) also reported difficulty in identifying first graders' solution processes and focused on problems in which children could externalize their method of solution (problems presented with objects). Attempts to determine <u>why</u>

a child chose a particular strategy or used a given operation in the computational process have often been unsuccessful (e.g., Zweng, 1979). Thus, two critical aspects of the individual interview procedure are the choice of follow-up questions and the use of tasks that elicit solutions which are based on observable or easily inferable behaviors.

Opper (1977) pointed out some of the procedural difficulties associated with the individual interview method. Among these were (a) the possibility that the child would not be at ease and perform naturally in the course of dialogue with the interviewer, (b) the problem of the interviewer maintaining neutrality and avoiding attempts to elicit "correct" answers, (c) the misunderstanding of language not adjusted to the child's level, (d) insufficient time for the child to reflect on the problem and to develop his/her explanations, and (e) the interviewer's interpretation of the child's actions and responses on which subsequent questions are based.

One of the most serious problems with interview data is that children's explanations of how they solved a problem may not accurately reflect the processes that they actually used. The interview procedure may change how a child solves a problem, or children may have difficulty articulating the process that they really used and therefore describe another process that is easier to explain. Or they may try and second guess what they think the interviewer is looking for. Another serious problem is that the inferences drawn from an interview involve a great deal of subjective judgment on the part of the experimenter.

Because of these limitations, researchers have sought alternative procedures that do not rely on children's explanations and can be based

on more objective measures. One of the more popular techniques is the use of response latencies.

Response latency. Response latencies have been used for a number of years to assess problem difficulty (Arnett, 1905; Knight & Behrens, 1928; Smith, 1921). More recent studies employing the response latency methodology have grown out of Suppes' initial work in this area (Suppes & Groen, 1966, 1967; Suppes, Jerman, & Brian, 1968). These studies have used response latency as a method for investigating possible solution strategies rather than as a relative difficulty indicator for addition and subtraction patterns, and have hypothesized various counting strategies. Reaction times are assumed to be a linear function of the number of counting steps required to solve a problem, and linear regression analysis is used to identify which counting model best fits the observed latencies. For a more complete description of the response latency paradigm, see Suppes and Groen (1967) and Groen and Parkman (1972).

The performance models used in the response latency studies hypothesize the existence of certain mental operations, i.e., setting a counter to a value and incrementing (or decrementing, or both) that counter by one. The five models hypothesized for addition problems of the form a + b = ☐ are the following:

(1) The counter is set to 0, it is incremented by a units, then it is further incremented by b units. (ALL)

(2) The counter is set to a (the left-most number), and is incremented by b units. (LEFT)

(3) The counter is set to b (the right-most number), and is incremented by a units. (RIGHT)

(4) The counter is set to the minimum of a and b, and is incremented by the maximum of a and b. (MAX)

(5) The counter is set to the maximum of a and b, and is incremented by the minimum. (MIN)

The companion models for subtraction problems of the form c - b = □ (or items with a subtractive structure such as the missing-addend items of the form a + □ = c or □ + b = c) are the following:

(1) The counter is set to 0, it is incremented c times, then it is decremented b times. The solution is the final value of the counter. (0, INC, DEC)

(2) The counter is set to c and is decremented b times. The solution is the final value of the counter. (DEC)

(3) The counter is set to b and is incremented until c is reached. The solution is the number of times the counter has been incremented. (INC)

(4) The counter is set to 0, it is incremented b times, and is then incremented until c is reached. The solution is the number of times the counter has been incremented during the second phase. (0, INC, INC)

(5) Either method (2) or method (3) is used, depending on which involves fewer increments of the counter. (FEWER)

Two data reduction methods have been used in response latency work. One involves an analysis by subject, in which the various counting models are fit to each subject's performance data (e.g., Groen, 1968; Groen & Resnick, 1977; Rosenthal, 1975). The other involves an analysis by item, in which a mean latency is computed for each item across all subjects

(e.g., Groen & Poll, 1973; Jerman, 1970; Parkman & Goren, 1971; Suppes & Groen, 1967; Svenson & Broquist, 1975; Woods, Resnick, & Groen, 1975). Svenson (1975) employed both methods of analysis, and Buckley (1975), and Winkelman and Schmidt (1974) used median rather than mean latencies. A great deal of information is lost in using latency averages which may hide important variability. This suggests that fitting individual subjects to regression models is the more appropriate procedure.

The assumptions of the response latency paradigm are presented in Suppes and Groen (1966, pp. 5-6), and are more succinctly stated in Rosenthal (1975, pp. 45-48) and in Svenson (1975). One assumption is that a constant time is required for each increment of the counter. A second is that a constant time is required to set the counter to its initial value. These two are independent of each other and independent of the values in the problem in which they are used. The time required to determine whether enough counts (increments) have taken place is "assumed to be a random variable independent of how many steps the counter should move" (Svenson, 1975, p. 297). This means that the time required for the decision to continue counting is constant, regardless of how close one is to finishing.

One assumption of the latency model which remains suspect with the time required to decide which of two values given in the item is larger. The time is assumed to be constant despite substantial evidence to the contrary. Moyer and Landauer (1967), Aiken and Williams (1968), and Restle (1970) identified factors that affected the speed with which pairs of numbers were compared. Among these are left or right position of the

larger number, size of the larger number, and the difference between the larger and the smaller number.

Groen and Parkman (1972) dismissed this evidence. They argued that "the process of finding the largest and smallest addends has only a second-order effect on the reaction times" (p. 332), and that the amount of time in question is on the order of only 10 milliseconds. They also contended that Moyer and Landauer's (1967) procedure indicated a strong relationship between the absolute difference of the addends and the magnitude of the smallest of the two addends. Groen and Parkman suggest that to determine the minimum addend "the subject searches (or generates) the discrete number line, beginning with one and continuing in increments of one until he finds one of the two numbers" (p. 332). No conscious counterpart of this process has been identified.

Another drawback of the response latency paradigm is that extreme values may result in a reasonably good fit of latency data to the regression equation of a particular strategy. This can give the impression that children consistently use a given strategy, whereas only a piece of the data really fits the model well. For example, a reanalysis of the data from a study by Groen and Parkman (1972) indicated that their best fitting model was much more appropriate for certain number domains than for others (Siegler & Robinson, 1981).

Analysis of errors. The third technique that has been used to infer children's solution processes is the analysis of error patterns (Lindvall & Ibarra, 1980a; Riley, Greeno, & Heller, in press). For simple addition problems involving basic facts, it is usually difficult to infer what strategies may be implied by errors to specific problems,

as can be done for algorithmic solutions to multi-digit problems (Brown & Van Lehn, 1982). However, certain general solution strategies will allow students to solve some problems but not others. By examining the patterns of errors over groups of problems, inferences may be drawn regarding strategies used. Certain errors also result in answers that consistently relate to problems in a predictable way. By observing a consistent pattern of errors, one may infer that the errors are caused by a particular incorrect procedure. Certain counting procedures may result in systematic errors, but unless the particular error is observed over a number of cases, it is not possible to determine whether a systematic error or a random counting error has occurred. An example of a systematic counting error is one in which a child counts on from a given number to find a sum but instead of counting on the correct number of places beyond the given number, the child includes the given number in the count. For example, to add 5 + 3, the child counts 5, 6, 7 and responds that the answer is 7. This is a systematic error that will consistently result in an answer one less than the correct answer. If a child consistently responds with an answer one less than the correct answer, one can infer that the child may be committing this error. However, caution is required in concluding that a child has used a particular strategy as other "buggy" procedures may result in the same incorrect answer.

Although certain errors, like adding when subtraction is called for, are relatively easy to detect--many errors are difficult to identify, especially when several errors occur in combination. Analysis of errors

also provides relatively little information about correct strategies. Thus, although error analysis provides only limited information on the strategies children use, a careful analysis of errors does help to round out the picture since response latency data is usually most effective in choosing between correct strategies.

Results of Latency Studies

Response latency studies have been carried out with subjects of all ages. For the purpose of this discussion, they have been divided into three groups determined by the general age range of the subjects, since it is likely that the processes used by subjects change over time. The studies with young children (preschool through first grade) used subjects having little or no formal (school) mathematics experience with addition and subtraction. A second group of studies used elementary or middle school students who had experienced extended instruction in addition and subtraction. The third group of studies was carried out with adult subjects whose strategies presumably represented mature approaches to the problems.

Studies with young children. Several studies utilizing the response latency methodology have used young children as subjects. Suppes and Groen (1967), Groen (1968), and Groen and Poll (1973) hypothesized certain strategies which could be used by first graders on simple addition and subtraction problems.

Suppes and Groen (1967) presented the first test of the five addition models discussed above. Thirty first graders were tested during the first half of the school year on 21 items presented in the form a + b = □,

with sums less than 6. Two models, ALL and MIN (see the list of addition models given above) were found to fit the data, with the MIN (count from larger) model providing the best fit. Groen (1967, 1968) extended the Suppes and Groen study to the larger class of addition items having sums less than 10. The 37 subjects in this study, had completed most of first grade. For 20 of these children, the MIN model was the only one which fit their performance. Only one of the remaining 17 subjects fit any of the other models. As in the Suppes and Groen study, problems involving doubles had consistently lower response latencies than other items with the same minimum addend. These lower latencies on problems involving doubles are explained by assuming that a reproductive (memory) process rather than a reconstructive (counting) process is used in these problems. Groen and Parkman (1972) noted that Groen's (1967) estimate of the slope of the regression line for the MIN model was nearly equal to the estimate of the speed of silent counting given in Beckwith and Restle (1966) and Landauer (1962). This provides additional support for the inference that children were indeed using a Counting On From Larger strategy to solve these problems. Replications and extensions of these findings with older subjects are discussed below.

The Groen and Poll (1973) study included first graders in the first half of the school year who had been taught addition but not subtraction. Their response latencies on missing addend sentences of the form $a + \square = c$ and $\square + b = c$ was tested against the DEC, INC, and FEWER models listed above. The FEWER model was the only one which fit the observed latencies and then only for half of the items, those with the

placeholder in the b position. Problems involving doubles again yielded uniformly lower latencies.

None of the above studies controlled instruction, so they provide no clear evidence whether the hypothesized counting strategies are invented or learned through instruction. To examine this question, Groen and Resnick (1977) taught the Count All (ALL) strategy for symbolic addition problems with addends less than 6, and sums less than 10, to preschoolers who were proficient at counting but who had not been exposed to instruction in addition. Items were presented in a + b form, and blocks were initially manipulated to demonstrate the Counting All strategy. After being instructed in the Counting All strategy only, and given extended practice, the performance of 5 of the 10 subjects best fit a Counting On From Larger (MIN) strategy which was presumably invented by the children themselves. Groen and Resnick noted that these preschool subjects followed a progression from the use of blocks to the use of finger counting to the use of mental counting strategies. They contented that the immediate representation of addends by the fingers of one hand is a physical analog of and precursor to the mental operation of setting a counter to a nonzero starting value.

The response latency studies with young children suggest that before children have much formal instruction on the addition and subtraction operations, their performance can be modeled by strategies which involve counting. Furthermore, these counting strategies are efficient ones that involve the fewest steps and are constructed by the children independent of instruction.

Studies with older children. Response latency studies with older children extend the findings of those using young children. Three studies with subjects ranging from third through seventh grade (Jerman, 1970; Svenson, 1975; Svenson & Broquist, 1975) investigated performance on subsets of the 100 basic addition facts. In all three studies, the MIN model provided the best description of performance. When used by Jerman, the MIN model included the assumption that for items in which the smaller addend appeared first, a transformation to the commuted form would take place prior to incrementing of the counter. Svenson also reported that latencies for items with the smaller addend first averaged 0.1 seconds greater than those for the same addends with the larger addend first. Problems involving doubles again resulted in low latencies in each of these studies.

Woods et al. (1975) compared the response latencies of second and fourth grade subjects on items of the form $a - b = \square$, $a \leq 9$. All 20 of the fourth graders and 30 of the 40 second graders were best fit by the FEWER model. Among the remaining second grade subjects only 2 were fit by INC, and 6 by the DEC model. These data suggest that with age, children progress from less efficient to more efficient processes for solving the canonical subtraction items just as they do for addition items. Also, as in the addition studies, doubles (and inverse doubles) were easier than other problems with comparable minimum addends.

One conclusion suggested by the Woods et al. study is that the FEWER model, which was used by the majority of third and fourth grade subjects to solve subtraction problems, was the same as that identified

for 7-9 year olds on missing addend problems of the form a + □ = c (Groen & Poll, 1973). This suggests that older children solve these two types of symbolic problems by the same process, but use a different process to solve open sentence problems of the form □ + b = c (Groen & Poll, 1973; Rosenthal, 1975). Rosenthal found more overall support for the use of trial (successive substitution) models than for counting models on these items.

A second conclusion suggested by the Woods et al. study concerns the procedure by which the most efficient process is chosen when the FEWER model is used. If subjects do use a process in which they must decide which of INC or DEC is more efficient, the question arises as to how such a decision is made. Groen and Poll (1973) offered two reasonable ways in which a subject might decide whether to increment or decrement. These are (a) making a rough approximation and (b) having sufficient familiarity with specific instances to "know" what to do. The latter would be based on a search of an incrementing list and a decrementing list stored in memory. A remote, but conceivable alternative to these is that of simultaneous incrementing and decrementing until one generates an answer. Woods et al. provided an additional explanation of how this choice might be made. Based on the accounts of two subjects, they suggested that twice the subtrahend is compared to the minuend, and that when the minuend is greater, decrementing is chosen. Otherwise incrementing is more efficient.

The response latency studies with older children who have more extensive school mathematics experience confirm many of the findings of the

studies with young children. They indicate that even though there is a definite trend toward the use of more efficient procedures by older subjects, counting strategies for solving simple addition and subtraction problems persist throughout the elementary school years.

Studies with adults. Response latencies have also been used to test the above counting models against adult performance. Using six subjects and the 100 basic addition facts, Parkman and Groen (1971) found the SUM and MIN models to nearly equally fit the pattern of latencies. They also supported Svenson (1975) in finding that 29 of 44 item pairs had longer latencies when the smaller addend appeared first. Use of a derived number fact based on 10, similar to that identified by Svenson, was found when one addend was 9. Consistent with the studies using children, lower latencies of doubles were attributed to use of a recall process.

Winkelman and Schmidt (1974) hypothesized that an association (recall) process could account for Parkman and Groen's (1971) data. Assuming that sums with larger addends are practiced less often, the strength of association between addends and sum would decrease, producing a monotone increase of reaction time with increasing minimum digit. (This monotone increasing function would fit fairly well with the linear structural model, MIN, proposed by Parkman and Groen.) Winkelman and Schmidt offered support for the existence of, but not necessarily exclusive use of such a recall process. With their six subjects, there were significantly more associative confusions (e.g., errors such as $3 + 4 = 12$, or $4 \times 5 = 9$) than nonassociative (unrelated) errors on

items using addends or factors of 3, 4, and 5, so they rejected the MIN model since it would predict no difference between associative and non-associative errors.

From three experiments with a total of 70 adults, Buckley (1975) concluded that a minimum of two processes--recall and counting--are needed to explain adult performance on addition and subtraction problems. He found a significantly higher correlation between the sum of the addends and response time than between the minimum addend and response time, and concluded that the SUM model was the best-fitting counting model. Contrary to Parkman and Groen (1971), Buckley also concluded that subjects do not compare addends and count on from the larger addend. He found no significant difference between the latencies of a group given larger addends first and a group given smaller addends first.

An analog, number line model has been proposed (Aiken, 1971; Restle, 1970) as an alternative to both the digital, counting model and the recall model for adult performance on simple addition problems. Aiken found response latencies proportional to the magnitude of both addends. This supports a model in which number lines corresponding to addends are mentally concatenated to form a sum.

Thomas (1963) presented another alternative to the counting models in which response latency is proportional to the log of the sum of the members of the number triple used in the addition or subtraction problem. For addition items of the form $a + b = c$, $\log (a + b + c) = \log 2(a + b) = \log (a + b) + .3010$, so the sum of the addends is the crucial value in this model. However, the relationship is not linear as in the SUM model.

In general, for response latency studies with adults, performance is not consistently accounted for by any of the hypothesized models. There is evidence that recall, counting, and derived fact processes may all be used, especially in light of the high variability of response times for an item by a given subject (e.g., Buckley, 1975). Some parallels were found between the studies with adults and those with children, yet no clear picture of adult-level, mature processes is given by the response latency work. The lack of a definitive description of processes used by adults may be due in part to the lack of attention paid to describing the recall process with models similar to those describing the counting processes. It seems reasonable to assume that many addition and subtraction facts are available to adults via some form of direct recall rather than by a reconstructive counting procedure, but the response latency studies have done little to substantiate or refute this conjecture.

Recall. While nearly all of the response latency studies acknowledge the possibility of some type of recall (memory, reproductive) process being used for certain problems, only Jerman (1970) has attempted to formulate and test a specific model of recall performance. Jerman's model assumed that addition facts are stored as elements of a mental array in which the subject proceeds from (0,0) to the coordinate (a,b) for the fact $a + b$. Thus, a value of $\sqrt{a + b}$ is associated with the "shortest route" to that combination. This means that $4 + 5$ and $5 + 4$ should be of equal difficulty but would have their values located at different points of the array. This type of recall model assumes a type of sequential

access to facts rather than random access. In order to retrieve the larger facts, the subject must "pass by" the smaller ones. This seems to be a somewhat restrictive assumption.

Groen and Parkman (1972) discussed various recall models. They suggested that "the counting models could easily be reformulated as retrieval algorithms that calculated an index, rather than a sum, with the index being used for a memory retrieval operation" (p. 342). Another formulation given is a list structure in which the setting operation corresponds to accessing a list, and the incrementing operation corresponds to finding the next element in the list. They cited the similarity between the memory search rate found by Sternberg (1969) and the incrementing rate used by adults on additions as evidence that such a list-based recall process might exist (p. 340).

Svenson, Hedenborg, and Lingman (1976) noted that

> if a certain proportion of the answers were directly retrieved from LTM, and thus not reproduced through the steps in the model . . . this could not be detected in a regression analysis of solution times. Only when a great majority of the subjects in most of the cases had the answer to a given problem stored in LTM, was it possible to detect this fact in the earlier presented analyses of latencies. The only additions fulfilling this requirement were the ties. (p. 169)

They have admitted here that regression analysis of response latencies is an inadequate determinant of whether or not recall is being used.

Winkerman and Schmidt (1974) contended that latency of recall might be proportional to the size of the addends (or size of the minimum addend). They attributed this to the hypothesis that "larger sums are practiced less often" (p. 734), this hypothesis was tested by Groen (1974,

pp. 8-9). Twelve 4 year olds were taught addition facts with sums less than 10 using two different methods of instruction--one group was given more facts having larger sums and the other group was given more smaller sum items. Items with small sums were recalled fastest in both groups (after 11 practice sessions). Groen cited this result as support for the hypothesis that the children were actually learning a counting algorithm rather than the effect being due to differential practice.

Information processing. The response latency studies raise questions about cognitive information processing mechanisms. Most of the studies hypothesize the existence of a single counter which is set and then incremented or decremented, one unit at a time. The question arises whether an additional counter is necessary to keep track of the number of times the counter which contains the total (or result) has been incremented. Several of the studies suggest that simultaneous operation of two counters may be a more reasonable model. Buckley (1975) suggested that both the MIN and the ALL models require the existence of two counters which are "yoked," counting together (with alternate rather than simultaneous incrementation) so that one records the number of increments while the other records the successive values terminating in the result (pp. 7-9). Beattie (1979) identified counting strategies which mirror this "yoked" counter situation (e.g., for 14 - 6, the subject says, "7, 1; 8, 2; 9, 3; . . . 14, 8").

Rosenthal (1975) also discussed the use of two counters in conjunction with his trial value models. In these models, a trial value is chosen and entered into a counter. That trial value is then validated

by one of the addition or subtraction counting models. If the trial value is incorrect, the trial counter is incremented or decremented and the new trial value is then validated.

Groen and Parkman (1972) raised the issue of two counters but stated that the counter which records the number of increments "can be assumed to influence each of the five counting algorithms in a uniform fashion, and since it is difficult to conceive of a way in which this operation could affect the linearity of the predictions, it is not discussed any further" (p. 331).

Svenson (1975) discussed information processing requirements of the response latency counting models in terms of the number of items to be operated on in STM. He contended that "three entities may be operated on in short-term memory: actual value of the counter, the unit that is presently added to the counter setting, and the number of units remaining to be counted" (p. 300).

Error Analysis

Strategies used to solve addition and subtraction problems can be inferred from errors in several ways. One is to hypothesize general strategies that will allow children to solve some problems but not others, and then match their performance with predicted patterns. Another attempt to classify errors for specific problems, and infers that certain strategies have resulted in specific types of errors. No studies so far have systematically attempted to characterize children's strategies for abstract problems based on analyzing patterns of correct and incorrect responses, although inferences may be drawn from the studies reviewed

earlier that compare the difficulty of different types of problems. For example, the research cited earlier in this review consistently found that open sentence problems of the type $\square + b = c$ were significantly more difficult than any of the other five types. Since the initial quantity is unknown, this problem would be difficult for any child who used a strategy that started with the initial quantity presented in the problem. These results would be consistent with predictions based on such a strategy.

Several studies have attempted to characterize errors on basic fact addition and subtraction problems. The classification schemes are very general or require a high level of inference. In general, these studies provide only limited insight into children's strategies.

Beattie and Deichmann (1972) classified errors on addition and subtraction open sentences as: basic fact (computation), incorrect operation (process), and unclassifiable (random). The generality of this classification scheme virtually precludes making conjectures regarding the type of strategy used.

Thyne (1941) analyzed Scottish Primary Division students' errors on basic addition and subtraction facts presented in $a + b = \square$ format. He found that for basic addition and subtraction facts, 11% and 15% of the errors, respectively, deviated by 7 from the correct answer, and that the wrong operation was performed 15% and 21% of the time. Responding with a given number occurred on 10% and 17% of the items, respectively. Other classifiable errors involved (a) zero--27% and 5%, (b) giving a "sequence" response such as 5 + 7 = 8--9% and 11%, and

(c) incorrect use of doubles such as 5 + 8 = 16--7% and 0%, respectively. For both addition and subtraction, findings showed fewer than 15% of the responses to be unclassifiable.

In the above studies, interpretation errors involved use of the wrong operation and procedural errors involved recall or computational (counting) errors with basic facts, as well as other unclassifiable errors. Finer analysis of these "other" procedural errors suggests that a variety of systematic errors may be present. Thyne's (1941) analysis provides some support for the inference that a variety of strategies are employed.

Direct Observation of Strategies

A variety of solution strategies have been observed among subjects of various ages asked to solve symbolically presented addition and subtraction problems in an individual interview setting. In fact, many of the strategies identified by direct observation have not been reported at all in studies using high-inference techniques. In the section which follows, the range of strategies which children and adults use when solving simple symbolic addition and subtraction problems is described. That is followed by a review of the more systematic research on children's problem-solving strategies.

Identification of strategies. Over 75 years ago, Arnett (1905) and Browne (1906) identified a variety of strategies used by adults to solve simple addition problems. Browne noted that counting by one and counting by two were used by eight adults on single-digit addition items. Arnett and Browne also identified the practice of adding the smaller

digit to the larger even if the smaller was the first addend. In column addition, some subjects frequently reordered the addends to add 10 complements or other easily assessable sums.

Ginsburg (1975, 1976, 1977b) contended that counting forms the core of children's practical arithmetic and that early solutions to addition and subtraction problems involve counting strategies. Later, more efficient strategies evolve which are based either on more sophisticated counting techniques or on a core of known facts. Ginsburg's case study analyses have provided ample evidence that such strategies are used. He identified strategies such as Counting All, Counting From the Larger or Counting From the Smaller addend, and the use of a core of known facts which involve doubles or ten to derive other facts.

Riess (1943a) identified many of the counting strategies used by preschool and early elementary students, and also distinguished between those used in the presence or absence of manipulatives. Helseth (1927) found that adults used dots, fingers, and other strategies based on a core of known facts to solve combinations they determined to be difficult. McLaughlin (1935) tested 125 three-, four-, and five-year-olds on abstract addition tasks presented with concrete, pictorial, or no aids. The data was summarized in very general descriptive terms, but it appears that typical performance ranged from inability of the three-year-olds to grasp the task, to use of Counting All and occasional Counting On among four-year-olds, and to Counting All, Counting On and some occasional use of facts with the five-year-olds. Presence of manipulatives improved performance.

Peck and Jencks (1976) administered missing addend problems in abstract form to first graders and reported two strategies used: overt or mental counting and recall of facts. Of 103 subjects who could solve missing addend problems, 80 were observed as counting, a "few" counted mentally, and 17 used recall. They noted that those students who used counting strategies were able to extend their strategy to solve similar problems presented concretely, and to solve more complicated multiplication problems and concretely represented items of the form ax + b - c. "The students who answered from memory were unable to solve subsequent problems. . . . In fact, they were unable to solve problems they had answered correctly in symbolic form when these same (or similar) problems were posed with concrete physical materials" (p. 659).

Thornton (1978) has documented the use of various taught and untaught counting strategies as well as strategies involving derived number facts among second and fourth graders. Smith (1921) similarly identified subjects who use derived facts on addition problems. Since his subjects were in the third through seventh grades, Smith considered these "roundabout schemes" to be a handicap for them, and cautioned against allowing lower grade pupils to use such strategies.

Beattie (1979) interviewed 98 fifth and sixth graders and found that many of the strategies listed above were used when basic subtraction facts could not be recalled. The derived facts noted by Beattie include the use of doubles, the use of 10 for facts involving 9, and use of 10 as a bridge, the use of any known fact followed by sequential

generation of related facts, and regrouping of the minuend as (10 + x) when a two-digit minuend is given.

Flournoy (1957) examined strategies used by third graders on "higher decade" (i.e., two-digit plus one-digit) addition problems. She found a variety of strategies similar to those identified above for addition of basic facts. An important finding in this study was that nearly 20% of these third graders used several different methods within a 12-item test.

Rosenthal (1975) attempted to get 9-year-old subjects to verbalize about their solutions to noncanonical addition and subtraction open sentences. Three of 25 subjects explained that they had used a systematic substitution (trial and error) strategy, but "most subjects were unable to provide any explanation at all" (p. 83).

Lankford (1974) interviewed 176 seventh graders and found that counting was the most frequently used strategy for whole number additions. Approximately 25% of these seventh graders used fingers and another 16% used marks or motions when counting. Many derived facts involving doubles were also identified. Lankford emphasized that even at the seventh grade level, "pupils vary widely in the computational strategies they employ in operations with whole numbers" (p. 29).

Analysis of strategies. The studies reviewed above indicate the wide variety of strategies used to solve addition and subtraction problems involving small numbers, but many of them were not carefully documented, so they can provide only a general impression of the actual range of strategies used. The following studies were more systematic in their approach. Once again the subjects' age level provides the most convenient means of aggregating the studies.

Davydov and Andronov (1981) interviewed 220 children aged four through seven on addition problems that were presented either abstractly or with one addend concretely presented. They identified three levels of strategies and described variations within each of these levels. The strategies which Davydov and Andronov identified for addition are a Counting All strategy and two variations of Counting On. The Counting All Strategy was observed in 14% of their subjects and involved the use of objects to model the addends and point-counting to enumerate the final set of objects.

The second level, which included 20% of the subjects, was characterized by counting without objects. Although objects were not used, children at this level found it necessary to count from 1 ("1, 2, 3, 4--5, 6" for a problem such as 4 + 2). Davydov and Andronov identified a number of hand and body movements which accompanied this strategy. They attributed these movements to a mental reconstruction of the objects which is assumed to take place as the child moves from actions with objects to action with "assumed sets."

The third level involved Counting On with the first addend considered as a unit, e.g., "4--5, 6" for the problem 4 + 2. Counting On was used by 55% of their sample. Generally, some type of sweeping motion was made to acknowledge the objects of the first addend, and the number word representing the first set was uttered in a drawn-out manner.

Davydov and Andronov also identified an erroneous strategy which they referred to as "imaginary adding-on." With this strategy, the child touched one object of the set and designated it by a number word corres-

ponding to the quantity of objects constituting the first addend. But these children were unable to distinguish between the number assigned to a set and the labeling of an element in counting. Upon further examination, this limitation led to a number of errors in attempting to count on.

Houlihan and Ginsburg (1981) presented three types of symbolic addition problems (one-digit plus one-digit, one-digit plus two-digit, and two-digit plus two-digit) to first and second graders. They found that counting constituted the predominant strategy among first graders and that approximately equal numbers of subjects used Counting All and Counting On strategies. Counting All was used only infrequently by second graders, but Counting On constituted nearly half of the appropriate strategies; number facts and derived facts comprised the remaining half. The use of appropriate strategies ranged from 77% on one-digit items to 4% on two-digit items for first graders and from 91% to 45% for second graders.

Houlihan and Ginsburg presented problems orally and in written form but found no difference between these presentation modes for symbolic items. Also, paper and pencil and manipulatives were available, but the manipulatives were seldom used by second graders. Manipulatives were used with approximately half of the first graders' counting strategies.

Brownell (1928) observed the strategies of 14 children aged seven to nine on each of 14 single-digit addition problems with sums greater than seven. Six had the larger addend first, six had the smaller first, and two were doubles. He identified counting, derived facts, and recall of basic facts as the primary methods of solution, and suggested that counting

formed the lowest stage of development for addition and "meaningful habituation," i.e., recall with understanding of number relationships. Seven of the 14 subjects used recall as their predominant strategy, three children used recall and derived facts with approximately equal frequencies, two children primarily used counting, and two children frequently used all three strategies given above.

Brownell also provided data regarding the consistency with which students added up or down for vertically presented items. Four of 14 students consistently added in one direction, while the other 10 varied the order in which addends were combined. Questioning revealed that this was done to achieve the "preferred form" of a combination, e.g., 7 + 3 as opposed to 3 + 7. For example, 9 + 6 was much preferred to 6 + 9. Brownell did not explicitly describe whether the preferred form always involved the larger addend first, but the implication is that this was the case. This might be taken as evidence for the behavior of transforming the addends so that a Counting On From Larger strategy could be used.

Brownell (1941) conducted a longitudinal investigation of 40 first graders' and 60 second graders' strategies on abstract addition and subtraction problems. He individually interviewed children using both items which had been taught and items which were unfamiliar. These interviews were carried out at midyear and at the end of one school year. Brownell tabulated strategies in six categories: no attempt, guessing, counting from 1, partial counting, solution, and recall of number facts. Brownell's

solution category includes derived facts based on doubles, use of a known fact, use of addition facts for subtraction problems and vice versa, and use of a fact in commuted form, e.g., using 3 + 1 = 4 for 1 + 3 = □. Several of Brownell's results are noteworthy. Recall of facts was the predominant strategy at all levels; partial counting was more frequently used by second graders than first graders; counting strategies occurred infrequently in comparison to guessing, solution, and recall of number facts; and more than 20% of the second graders' strategies on subtraction problems were classified in the solution category.

Russell (1977) observed 32 third graders' strategies on 15 verbally presented symbolic addition items with sums between 19 and 48 and between 100 and 500. Concretely presented items (dots on cards) were also included, and paper and pencil and manipulatives were made available. Russell found that "third graders use strategies that are appropriate to individual problems" (p. 157). Counting All was the most frequent strategy with the dot items, while the written algorithm was used on at least 90% of the large addend items and on those for which a written solution was required. Counting On and derived facts were each used on about 15% of the items with sums less than 48, while the standard algorithm accounted for 63% of the strategies on those items. Russell concluded that a "trend towards economy and efficiency" (p. 158) exists at the third-grade level.

Grouws (1972a, 1974) observed 32 third graders' strategies for solving both types of noncanonical addition and subtraction sentences with sums less than 19 and between 41 and 100. The strategies he identified included counting, trial and error, use of facts, use of a written algorithm, use of derived facts, and transformation to an equivalent sentence.

Although items were presented in two contexts, symbolic and symbolic with verbal problem, "there was a close similarity in pattern and in number of methods used in these two contexts" (Grouws, 1972a, p. 135).

As in Russell's (1977) study, counting strategies were used on approximately 15% of Grouws' items (for both basic fact and two-digit items). Derived facts (8%) and written algorithms (38%) were used somewhat less frequently than in Russell's study. The primary difference between Russell's findings and Grouws' was the use of facts when possible (17%) and the use of trial and error (7%). Transformation to an equivalent sentence occurred infrequently (2%). Facts and derived fact strategies were used more frequently on addition sentences, and standard algorithms were used more often on subtraction sentences, although Grouws (1972a) considered these differences too small to warrant any generalizations.

Grouws' study with noncanonical forms was consistent with Russell's study in that third graders used strategies appropriate for the item (e.g., recall with small addends, algorithms with larger addends). A variety of strategies, including counting, continued to be used at the third-grade level. However, a trend toward the use of more efficient strategies was evidenced by the fact that approximately 55% of the strategies on noncanonical items involved recall or use of standard algorithms.

In a sequence of studies (Svenson, 1975; Svenson & Broquist, 1975; Svenson, Hedenborg, & Lingman, 1976), Svenson examined the strategies used by 9-12-year-olds on canonical addition items with sums less than 14. In the first study, following several sessions in which response latencies were measured, subjects were asked what strategies they would

use when answering a selection of 5-8 addition items. Counting From Larger, use of derived facts involving doubles or 10, and recall were the strategies identified. Svenson and Broquist (1975) asked similar questions and found that 22 of 26 subjects counted on from the larger addend on some items.

Svenson et al. (1976) interviewed eight subjects who were asked to tell "how they thought when they got the answer" to each of 250 items, excluding doubles and those with addends of 0 or 1.

> On the average about 74% of the analysed additions were solved by applying a consistent strategy. There were great individual variations, from very consistent subjects with about 90% of the answers in the same category, to subjects with only about half or one-third of the problems reported to be solved in the same way during four or five presentations in the experiment. (p. 167)

Recall and Counting On (from larger or smaller) each occurred on 36% of the items. Counting in steps greater than one occurred on 16% of the items, but this included responses like "8, then 9 and 11," for which the counting did not continue sufficiently to determine if the child was really counting in units of two. Derived facts involving doubles were used on 8% of the items, and other derived facts accounted for 4% of the strategies. Svenson et al. concluded that subjects "used highly individual methods for solving some of the problems" (p. 169).

Although longitudinal data on the development of strategies for addition and subtraction is sparse, Ilg and Ames (1951) reported data on strategies used by 30 children, followed from age five through nine. They concluded that addition strategies follow a sequence of development toward efficiency as reflected by Counting All, Counting From the Smaller Number, Counting From the Larger Number, and use of derived and known

facts. Subtraction strategies were similarly modified. Children first counted from 1 to the larger number, then down from it a number of counts equal to the smaller number. Later children began at the larger number and just counted down from it. Derived facts, often with doubles, Counting Up From Given Smaller Number, and the use of known facts were the latest appearing strategies. Ilg and Ames gave ages at which the above strategies were used, e.g., Counting All—age 5; Counting From the Smaller Number--age $5\frac{1}{2}$; Counting From the Larger Number--age 6; etc., but these ages must be interpreted in light of the fact that Ilg and Ames' sample was "somewhat above the average in intelligence" (p. 4). Nevertheless, their data confirms the progression toward increasingly efficient strategies which is suggested by the available cross-sectional data.

Conclusion. Children use a reasonably well-defined set of strategies to solve addition and subtraction problems involving basic facts. There appears to be an evolution from more primitive counting strategies to more efficient counting strategies, and finally to strategies based upon recall of number facts, but children at all levels of elementary and middle school continue to use basic counting strategies on symbolic addition and subtraction problems. It appears that many students do not commit all of the basic addition and subtraction facts to memory, nor do they at any level adopt a universal strategy for solving these items.

WORD PROBLEMS

Since there are a great many more variables upon which word problems are based than was the case for symbolic problems, the classification of research on word problems is more complex.

Nesher (1982) identified three main components that she contended account for the difficulty of verbal problems. These are

(1) the logical structure which includes the type of operation involved and the presence or absence of superfluous information;

(2) the semantic component which includes the contextual relationships contributing to problem structure and the verbal cue words included in the problem; and

(3) the syntactic component which includes structural variables concerned with the number of words, position of the component parts within the problem, etc.

In the analysis that follows, the logical structure dimension has been incorporated into the semantic component. Thus, there are two basic dimensions upon which word problems are described in this paper: one based on syntactic variables and the other based on the semantic structure of the problems.

Most of the research on syntax has attempted to account for differences in difficulty among problems, whereas most of the research investigating the effect of semantic structure provides either direct evidence of children's strategies or indirect support for the existence of hypothesized strategies. Consequently, rather than organizing studies on the basis of whether or not they investigated solution pro-

cesses, the syntactic-semantic distinction is used as a basis of organization. The review of word problem research is divided into three parts. The first part is not concerned with distinctions between problems but rather examines factors that affect children's overall performance on addition and subtraction word problems. The next two parts examine factors that account for differences in performance on different word problems. The first of these two sections is concerned with the syntactic component, and the final section is concerned with the semantic component. The first two sections focus primarily on problem difficulty; the third is concerned with solution strategies.

FACTORS RELATED TO WORD PROBLEMS

Knowledge of Basic Facts and Performance on Verbal Problems

Researchers have often suggested that computational ability correlates highly with problem-solving performance (Suydam & Weaver, 1975). Several studies have specifically addressed the relationship between computational skill with basic addition and subtraction facts and performance on addition and subtraction word problems.

Steffe (1970) reported a correlation of .49 between knowledge of addition facts and scores on a test of addition word problems. He noted that the most substantial correlation occurred for the lowest IQ group (.60) and the lowest group of quantitative comparisons ability (.68). He suggested that these children's difficulties with addition facts might be explained in large part by their inability to solve addition verbal problems in contrast to the curriculum's emphasis on learning facts at the expense of verbal problem instruction.

LeBlanc (1971) found a correlation of .39 between knowledge of subtraction facts and subtraction verbal problem performance. He regarded this as a relatively low correlation. In fact, correlations for groups of individuals of differing IQ and quantitative comparison levels were "judged to be so low that the relationship between children's knowledge of number facts and their performance on the problem-solving test is questionable" (p. 159).

Steffe, Spikes, and Hirstein (1976) found significant correlations between addition and subtraction fact scores when children were forced to solve problems without the aid of physical objects. When children were allowed to use manipulatives to help them solve the problems, however, only certain classes of problems were significantly correlated with knowledge of basic facts.

Additional support that problem-solving performance is related to knowledge of number facts is found in studies which have considered a range of number facts. Carpenter and Moser (1982) reported uniformly better performance on problems with sums between 5 and 9 than on problems with sums between 11 and 15. Vergnaud (1982) also reported uniformly higher performance when small constants were used than when one or both of the constants were between 12 and 15. Zweng (1979) concurred and reported that "for all grade levels [3, 4, 5, 6] and all ability groups decreasing the size of the numbers was very effective in helping students solve problems" (p. 61).

Overall, it is justifiable to conclude that children can solve addition and subtraction problems better when the constants in the problems are small numbers. It is also safe to conclude that knowledge

of basic facts is, to some degree, related to the ability to correctly solve addition and subtraction word problems. However, as LeBlanc (1971) noted, "Surely knowledge of basic facts is not sufficient for success in problem solving" (p. 159).

Presence of Aids

Much empirical attention has been focused on the effects of manipulative or pictorial aids on children's performance doing verbal addition and subtraction problems. The issue of the availability of aids is not a simple one. The conditions under which they were used have varied widely across studies. In some instances aids were merely made available to the subjects; in others the subjects were required to use the aids, and in yet others, the verbal problems were actually presented concretely, i.e., the problem data were presented via the experimenter's manipulation of the objects.

We will first consider studies in which manipulatives (objects and/or pictures) were available to the subjects. Bolduc (1970), Carpenter and Moser (1982), Gibb (1956), Ibarra and Lindvall (1979), and Schwartz (1969) have reported similar findings regarding the effect of manipulatives or pictorial aids on children's performance. In all five studies, the availability of aids resulted in improved performance. Gibb found that physical objects and pictures were about equal in effectiveness on three types of subtraction problems, and that performance of second graders was poorer when no aids were used. Bolduc reported that an absence of aids with first graders resulted in poorer performance than when two types of visual aids were used. Schwartz compared only pictorial

aids and no aids, and reported better performance when kindergartners were shown pictures of objects used in the verbal problems. Carpenter and Moser reported data on 144 first graders' performance on six addition and subtraction problems at three different times during the school year. For all six problems during each of the three interviews, when objects were available, performance exceeded or equaled that when objects were not available. In addition, the difference between performance with objects available and that with no objects available was more marked with problems in which sums were between 11 and 15 than with problems with smaller sums. Lindvall and Ibarra presented six verbal addition and subtraction problems to kindergartners under a variety of conditions. Two of these conditions were a verbal problem read by the experimenter without manipulatives and a similar reading with manipulatives available. The p-values for all six problems were 10-20 points higher when objects were available.

Steffe and Johnson (1971) also found uniformly higher performance with objects available. However, they found an interaction between the availability of objects and quantitative comparison ability. The facilitating effect of objects was less at the high level of quantitative comparison ability for certain addition problems. Steffe et al. (1976) and Hirstein (1979) reported the only negative evidence concerning use of manipulatives. They obtained p-values 10-30 points lower when objects were available. They conjectured that the superior performance often found when manipulatives are available might be a result of instructional practices which encourage the use of objects and discourage the use of fingers. They also suggested that children who are low on the scale of

being able to perform quantitative comparisons can still employ Counting All techniques with fingers to solve many verbal problems.

In some instances children have been required to use objects in their solutions of verbal problems, i.e., they have been instructed to "use the objects to solve the problem." Riley (1979) reported that kindergartners' performance "dropped significantly" in the absence of blocks, but first graders' performance was unchanged from verbal problems without manipulatives available to problems in which they were required to solve the problem by using the blocks.

Concrete presentation of the data and actions in the problem is the third condition in which aids have been used in verbal problems. Shchedrovitskii and Yakobson (1975) present a detailed logical analysis of the modeling and counting procedures a child must employ in order to solve problems which use sets of objects to present the problem. There is an essential difference between the use of objects in conjunction with a verbal problem, i.e., a condition in which the experimenter does not directly alter the condition of the manipulatives, and the use of objects as a vehicle for presenting the problem, i.e., a condition in which all or a portion of the manipulation is done by the experimenter. In the latter case, a portion of the child's task is eliminated; he or she must only be able to correctly solve the problem after it (or part of it) has been represented in the physical mode. Consequently, the performances exhibited in the studies below must be distinguished from those on tasks requiring the subject to solve the problem without assistance.

Lindvall and Ibarra (1979) used three variations of "concrete" presentation. In one, they used objects to show the sets described in the

problems; in another, they used a pictorial representation of the sets and the actions in the problem; and in the third, they used objects manipulated by the experimenter to model the sets and the actions in the problem. Their data indicated that the third condition resulted in drastically improved performance over the other two, and that performance levels under the first two were roughly comparable to performance when objects were available to the subject. In any case, all three variations of concrete presentation of verbal problems resulted in improved performance over that exhibited in the absence of objects.

Steffe (1970) and LeBlanc (1971) carried out similar investigations to compare first graders' performance on addition and subtraction problems, using (a) no aids, (b) concrete aids, and (c) pictorial aids. Steffe reported that "problems with no accompanying aids were significantly more difficult than either of the other two types of problems for all children involved" (p. 159), and LeBlanc concurred, noting that "the conclusion that the children solved the problems better with the presence of aids is well substantiated" (p. 150). LeBlanc also noted that the mean differences for problems with and without aids were greatest for children in the lower levels of IQ and quantitative comparison ability, although differences were not significant. Grunau's (1978) findings were consistent with those of Steffe and LeBlanc. She concluded that for kindergarten subjects the presence of aids was more crucial to successful performance among number nonconservers than among conservers.

Hebbeler (1977) studied prekindergartner's through second graders' performance on addition problems and found that when the problem data were presented concretely, more preschoolers could do the problems than

could without objects present, but that "the older children do not need the objects nor do they benefit from having them" (p. 113). In discussing her use of objects to present the problem data, Hebbeler noted that using objects in this way precluded the need for the subject to represent the quantities to be combined. These particular children seemed able to represent the data in the problems without assistance from the experimenter. The older children were performing near ceiling level; by second grade, her subjects with and without objects were correct on better than 93% of the items.

The method of concrete presentation in Starkey and Gelman (1982) and MacLatchy (1933) is somewhat different from that in the studies discussed above. MacLatchy and Starkey and Gelman presented problems by using objects to present the problem data but screened the first set of objects before any action was performed. These problems were not thereby reduced to simple enumeration. Children as young as three years old were able to solve some of the problems with small numbers presented in this fashion.

In another variation of concretely presented problems, Hendrickson (1979) had the subject model the first number in the problem before continuing to read the rest of the verbal problem, e.g., "Put 2 blocks in front of you. Now, if I give you 5 more, how many blocks will you have?" His p-values are lower than most of those for entering first graders who were given the entire problem in concrete form.

As early as 1933, MacLatchy documented better performance on verbal problems presented concretely than on verbal problems presented without aids. Similar findings have appeared many times since, whether the

manipulative objects or pictures have simply been available to the subjects or whether they have been manipulated by the experimenter to show the data and relationships in the problem. It is safe to conclude that the use of aids generally improves performance, but that this effect becomes less marked as the age of the subjects increases. One can also conclude that some of the difficulty in solving verbal problems stems from formulating a plan for solution of the problem, i.e., analysis of the relationships or actions, and the modeling of those relationships or actions; and the remaining difficulty relates to the mechanics of determining the result, once the problem has been represented. If the problem is presented using the materials, the first difficulty is removed. If the aids are simply available, they appear to aid in the representation of the problem and in the mechanics of finding an answer from this representation.

SYNTAX VARIABLES

A number of studies have investigated the effects of vocabulary or syntax on difficulty in verbal problems. The intent of these studies is to find structural variables within the problems which affect the success rate. A study which included both vocabulary and syntax as variables was reported by Linville (1970, 1976). His primary purpose was to ascertain whether or not either the degree of syntax complexity or the vocabulary level used in the problems significantly contributed to their difficulty when the computational operations required were kept constant. The syntax measure was a casual intuitive one apparently related to length of sentence and presence or absence of a relative clause. Both

syntax and vocabulary were divided into easy and hard, producing a 2 x 2 partitioning of the problems. Syntax and vocabulary level were both found to be significant determinants of difficulty in verbal arithmetic problems, with vocabulary level more important in determining success than syntax.

Blankenship and Lovitt (1976) manipulated vocabulary level in addition problems, using both accuracy and speed as dependent variables. One finding was that extraneous information impaired solution.

Several studies have discussed the role of various verbal factors in the solution of word problems. Nesher and Teubal (1975) investigated the effects of "cue" words on performance of addition and subtraction problems. They reported that the presence of certain distracting "key words" had a detrimental effect on first graders' choices of the correct operation in verbal problems.

Steffe (1967) interviewed 90 first graders to determine the effect of varying the types of objects described in combined addition problems. Half of the problems used objects with different names, e.g., "Someone had a marbles and b balls, how many toys did she have?", while the other half used objects with the same names, "Someone had a blue marbles and b green marbles, how many marbles did he have?" Steffe found that problems in which the objects had the same name were significantly easier than those with differently named objects. Bolduc (1970) found no significant difference on this factor, but Kellerhouse's (1975) replication of Steffe's study found significant differences in favor of problems with identically named objects for both first graders and second graders who solved problems using visual aids. Kellerhouse found no significant

differences between these two types of problems for second graders who had no visual aids available.

Steffe (1967) also compared first graders performance on problems with a quantifier, e.g.,

> John has some toys in his pockets. He has 5 jacks in one pocket and 2 marbles in another pocket. How many toys does he have in his pockets?

and similar problems without the quantifier,

> John has 5 jacks in one pocket and 2 marbles in another pocket. How many toys does he have in his pockets?

He found no difference in performance on the two types of problems.

Several studies have investigated the effects of varying the position of certain components of verbal problems. Rosenthal and Resnick (1974) varied the order in which temporal information was given in verbal problems modeled by $a + b = \square$, $a - b = \square$, $\square + b = c$, and $\square - b = c$. All problems given to the third graders involved action described in chronological order, e.g.,

> If Paul started out with 5 boats and he bought 3 boats, how many boats did he end up with?

or in reverse chronological order,

> How many boats did Paul end up with if he bought 3 boats and he started out with 5 boats?

They found that the reverse order was more difficult when percent correct was the criterion, but not when latency of response was the criterion. Bolduc (1970) found that the position of the question (before or after the data) was not a significant factor in difficulty of addition problems for first graders. This suggests that Rosenthal and Resnick's results may be due more to the temporal aspect than to the presence of the question in the first sentence of the problem.

Grunau (1978) tested whether verbal problems having a larger first addend were equal in difficulty to those in which the smaller addend was presented first. Grunau described as "tentative" her conclusion that problems in which the larger addend appears first are easier than those in which the smaller addend appears first. Conservers and nonconservers had significantly more correct responses when the larger addend was first, and transitional subjects exhibited a similar but nonsignificant difference. Vergnaud (1982), however, did not find that order of the data influences performance.

The most systematic research on word problem difficulty has involved the construction of linear regression models. The fundamental objective of this line of research has been to build comprehensive models to accurately predict the difficulty of different problems. This research attempts to identify components of the problems (both nonlinguistic and linguistic), and then to regress these variables multilinearly upon the observed difficulty level. "The term 'structural' indicates that the focus of attention is on the variables that characterize the specific problems themselves (e.g., the number of words in the problem) and on the variables that characterize the relationship between individual problems (e.g., the structural similarity of two adjacent problems" (Loftus & Suppes, 1972, p. 531). A further analysis of regression studies can be found in Barnett, Vos, and Sowder (1978).

In 1966, Suppes, Hyman, and Jerman wrote a paper analyzing the properties of various linear structural models and their possible uses in advancing a theoretical mathematics education. Their declared aim was to attach weights to various factors and then use them to predict the

relative difficulty or latency of response for a large number of items. This multilinear regression model, using factors suggested by an initially theoretical analysis of the problems, based on suggestions from other research, was repeatedly used by Suppes and his co-workers throughout the ensuing decade.

They believed that it would be possible to analyze and predict, with the use of meaningful variables, the response and latency performance of children solving arithmetical problems. Note that all variation was assumed to be inherent within the problem itself. The use of average observed success rates as the variable to be predicted, rather than individual responses, effectively ignored any differences among students. A widespread assumption among these regression studies is that a difficulty index can be established with small number of variables, specific to verbal problems, which will account for a large portion of the variance in observed success rates of such problems. No significant role is given to strategies used by the students; the concern is merely whether or not they correctly solved the problem.

Suppes, Loftus, and Jerman (1969) presented a series of word problems to 27 students via computer teletype. The variables were supposed to be "objective," with the result that they were all "count" variables: either 0, 1 (e.g., whether or not a problem required conversion of units), or ones which took on a finite set of values (e.g., length of problem as measured by number of words.) Sentence length was chosen as the most plausible variable associated with sentence difficulty. Approaches to this variable became increasingly complex in this series of studies. "In subsequent studies, we hope to look at the actual syntactic structure

of the sentences which should provide a more meaningful index of difficulty than mere word count alone" (Suppes et al., 1969, p. 4). In all, six structural variables were identified; and 45% of the total variance was accounted for, although none of the specifically linguistic variables were of major importance.

The next study in this sequence (Loftus & Suppes, 1972) incorporated a more complex measure of linguistic structure. One hundred word problems involving all four arithmetic operations were presented to 16 sixth grade students characterized as "of low ability." As well as the six old variables, two new ones were included. These were order (a 0,1 variable) which reflected whether or not the numerical data presented in an order other than one which could be used to solve the problem, and a measure of sentence complexity based on main word depth as described by Yngre (1960). With these eight variables, 70% of the variance could be accounted for, although the order of their relative importance in terms of which one entered the regression equation first was different from the previous study. Five variables: sequential, operations, depth, length, and conversion provided almost as good a fit.

Searle, Lorton, and Suppes (1974) continued this line of investigation, although their intention was to use the variables to structure a computer-based problem-solving curriculum. Twelve variables were initially used, although the depth measure was excluded, because of difficulties coding. An r^2 of .66 was obtained.

Jerman and Rees (1972) summarized the work done to that time. They discussed the evolution of the attempts and of the variables employed in the regression problem. "One of the purposes of these regression

studies has been to identify and quantify in a clear and explicit way a set of structural variables that account for a significant amount of the variance in the observed error rate. . . . What we hope for is to be able to generate sets of arithmetical problems of a specified difficulty level" (Jerman & Rees, 1972, p. 306). They summarized the mushrooming sets of variables being proposed in an attempt to account for ever-increasing proportions of the variance. They then attempt to fit the most robust variables to a new set of data on fifth grade students, where in contrast to the CAI format, the calculations were also being done by the student. After 9 variables, an r^2 of .59 was obtained. Following further adjustment and new less intuitive or easily implemented variables, an r^2 of .87 was obtained with only five variables.

Jerman (1973) attempted to replicate the above study using students of different grade levels (4, 5, and 6). He focused primarily on length, the simplest of the five variables successfully employed in the 1972 work, and the one with the oldest pedigree. This new study involved 19 variables, and length failed to consistently account for a significant amount of the variation. He concluded: "The task of further refining the definitions of the variables seems to be the next logical step. . . . Perhaps through a combination of structural and linguistic variables the ability of the model to predict error rate will improve significantly" (p. 122). Here, as in earlier works, his aim was to identify a small subset of variables (six or fewer) which could be used in further investigation. He was attempting to generate robust variables which might have explanatory power across grade levels. The focus on specifically linguistic factors was influenced by Krushinski (1973). Krushinski

investigated the influence of 14 specifically linguistic variables on the difficulty of verbal arithmetic problems for college students enrolled in an elementary mathematics methods course. Examples of his variables were sentence length, number of clauses, clause length, number of prepositional phrases, and number of words in the question sentence. These five were all found to be significantly related to problem difficulty and entered consistently into the regression equation.

The final study emerging from this group is Jerman and Mirman (1974). It is a major attempt to identify linguistic predictor variables. Seventy-three variables were identified and organized in seven different categories. These were measures of length (of problem), parts of speech, words, numbers, sentences, parts of sentences, and punctuation/symbols/characters. A study of 340 students in grades 4-9 was conducted to provide data upon which to regress these variables. A tremendously detailed analysis was included, but no discernable patterns were identified. Length was not significant at any grade level. One generalization they made was that linguistic factors are apparently better predictors of difficulty level for older students. They concluded, "At this point in the search for a set of linguistic structural variables which will account for the observed variance in proportion correct across grade levels [we seem] to have found that there is no such universal set" (p. 360).

SEMANTIC VARIABLES

The Semantic Classification of Word Problem Types

There have been a number of attempts to characterize basic semantic differences between various types of addition and subtraction problems.

One issue is how to classify a problem as either an addition or subtraction problem. Reckzeh (1956) pointed out that attempts have often been made to connect key words in verbal problems (more than, less than, etc.) with the operation of addition and subtraction, and he suggested a more desirable alternative. His alternative classified addition and subtraction verbal problems as involving additive or subtractive situations. He took issue with Van Engen's (1955) use of "additive or addition situation" to include missing addend problems modeled by the equations $a + \square = c$ and $\square + b = c$. Reckzeh argued that the structure of a problem is determined by the operation required to generate the solution. For Reckzeh, an additive situation is "one in which two or more groups of known size are joined to form a single group where the size of the latter group is to be determined" (p. 95).

Van Engen and Reckzeh arrived at different categories for addition and subtraction problems because they based their definitions on different dimensions. Kossov (1975) recognized this distinction and classified simple addition and subtraction verbal problems according to two aspects of problem structure which he referred to as Feature I and Feature II. The first feature of verbal addition and subtraction problems is based on the action described in the problem, either making larger or making smaller. The use of words such as "more" and "fewer" are the determinants of this feature. Kossov concedes that some problems may not involve action and thus may not have this feature present. The second essential feature of Kossov's classification is "the position of the unknown in the problem's structure" (p. 127). Problems can be classified along this dimension into two categories: Direct problems (those

with a canonical corresponding open sentence) and indirect problems (those with a noncanonical corresponding open sentence).

Addition and subtraction word problems have also been partitioned in several other ways. One approach distinguished between problems on the basis of whether or not action was involved (LeBlanc, 1971; Steffe, 1970). A second approach differentiated between problems in terms of the open sentences they represented (Grouws, 1972a, 1972b; Lindvall & Ibarra, 1980b; Rosenthal & Resnick, 1974). Both approaches overlook important differences between certain classes of problems. Recently, a group of researchers studying children's solutions to addition and subtraction problems have adopted a common framework which appears most productive in distinguishing important differences in how different problems are solved (Carpenter & Moser, in press; Riley, Greeno, & Heller, in press). This framework is generally consistent with several earlier classification schemes (Carpenter & Moser, 1982; Greeno, 1981; Nesher & Katriel, 1977) and incorporates the "take away," "joining," and "comparison" situations identified by Gibb (1956), Reckzeh (1956), and Van Engen (1949). This analysis proposes four broad classes of addition and subtraction problems: Change, Combine, Compare, and Equalize.

There are two basic types of Change problems both of which involve action. In Change/Join problems, there is an initial quantity and a direct or implied action that causes an increase in that quantity. For Change/Separate problems, a subset is removed from a given set. In both classes of problems, the change occurs over time. There is an initial condition at T_1 which is followed by a change occurring at T_2 which results in a final state at T_3.

Within both the Join and Separate classes, there are three distinct types of problems depending upon which quantity is unknown (see Table 2). For one type, the initial quantity and the magnitude of the change is given and the resultant quantity is the unknown. For the second, the initial quantity and the result of the change is given and the object is to find the magnitude of the change. In the third case, the initial quantity is the unknown.

Both Combine and Compare problems involve static relationships for which there is not direct or implied action. Combine problems involve the relationship existing among a particular set and its two, disjoint subsets. Two problem types exist: the two subsets are given and one is asked to find the size of their union; or one of the subsets and the union are given and the solver is asked to find the size of the other subset (see Table 2).

Compare problems involve the comparison of two distinct, disjoint sets. Since one set is compared to the other, it is possible to label one set the referent set and the other the compared set. The third entity in these problems is the difference, or amount by which the larger set exceeds the other. In this class of problems, any one of the three entities could be the unknown: the difference, the reference set, or the compared set. There is also the possibility of having the larger set be either the referent set of the compared set. Thus, there exist six different types of Compare problems (see Table 2).

The final class of problems, Equalize problems, are a hybrid of Compare and Change problems. There is the same sort of action as found in the Change problems, but it is based on the comparison of two disjoint

Table 2

Classification of Word Problems

Join	Separate
Change	
1. Connie had 5 marbles. Jim gave her 8 more marbles. How many marbles does Connie have altogether?	2. Connie had 13 marbles. She gave 5 marbles to Jim. How many marbles does she have left?
3. Connie has 5 marbles. How many more marbles does she need to have 13 marbles altogether?	4. Connie had 5 marbles. She gave some to Jim. Now she has 8 marbles left. How many marbles did Connie have to start with?
5. Connie had some marbles. Jim gave her 5 more marbles. Now she has 13. How many marbles did Connie have to start with?	6. Connie had some marbles. She gave 5 to Jim. Now she has 8 marbles left. How many marbles did Connie have to start with?
Combine	
7. Connie has 5 red marbles and 8 blue marbles. How many marbles does she have?	8. Connie has 13 marbles. Five are red and the rest are blue. How many blue marbles does Connie have?
Compare	
9. Connie has 13 marbles. Jim has 5 marbles. How many more marbles does Connie have than Jim?	10. Connie has 13 marbles. Jim has 5 marbles. How many fewer marbles does Jim have than Connie?
11. Jim has 5 marbles. Connie has 8 more marbles than Jim. How many marbles does Connie have?	12. Jim has 5 marbles. He has 8 fewer marbles than Connie. How marbles does Connie have?
13. Connie has 13 marbles. She has 5 more marbles than Jim. How many marbles does Jim have?	14. Connie has 13 marbles. Jim has 5 fewer marbles than Connie. How many marbles does Jim have?
Equalize	
15. Connie has 13 marbles. Jim has 5 marbles. How many marbles does Jim have to win to have as many marbles as Connie?	16. Connie has 13 marbles. Jim has 5 marbles. How many marbles does Connie have to lose to have as many marbles as Jim?

(Continued)

Table 2--continued

Equalize (Continued)	
Join	Separate
17. Jim has 5 marbles. If he wins 8 marbles, he will have the same number of marbles as Connie. How many marbles does Connie have?	18. Jim has 5 marbles. If Connie loses 8 marbles, she will have the same number of marbles as Jim. How many marbles does Connie have?
19. Connie has 13 marbles. If Jim wins 5 marbles, he will have the same number of marbles as Connie. How many marbles does Jim have?	20. Connie has 13 marbles. If she loses 5 marbles, she will have the same number of marbles as Jim. How many marbles does Jim have?

sets. Equalize problems are not as commonly found in the research literature or in most American mathematics programs; however, they do appear in the Developing Mathematical Processes (DMP) program (Romberg, Harvey, Moser, & Montgomery, 1974). These problems are also present in experimental programs developed in the Soviet Union (Davydov, 1982) and in Japan (Gimbayashi, 1969). As in the Compare problems, two disjoint sets are compared; then the question is posed, "What could be done to one of the sets to make it equal to the other?" If the action to be performed is on the smaller of the two sets, then it becomes an Equalize/Join problem. On the other hand, if the action to be performed is on the larger set, then Equalize/Separate problems result. As with comparison problems, the unknown can be varied to produce three distinct Equalize problems of each type (see Table 2).

The above analysis of addition and subtraction word problems is limited to simple problems that are appropriate for primary age children. It is not as complete as the framework proposed by Vergnaud (1982) that extends to operations on integers. Vergnaud's (Vergnaud, 1982; Vergnaud & Durand, 1976) classification of verbal addition and subtraction problems is consistent with the above analysis but encompasses a greater number of problem types. Vergnaud outlined six classes of problems which are generated by considering measures, transformations, and static relationships (relative states) as entities in the problem. His classification scheme included problems which involve:

(1) Composition of two measures (Combine problems) ("measure-measure-measure") M-M-M

(2) A transformation links two measures (Change problems) ("initial state-transformation-final state")

(3) A static relationship links two measures (Compare problems) ("measure-static relationship-measure")

(4) Composition for two transformations ("transformation-transformation-transformation")

(5) A transformation links two static relationships ("static relationship-transformation-static relationship")

(6) Composition of two static relationships ("static relationship-static relationship-static relationship")

Static relationships and transformations can be either positive or negative, i.e., they are represented by directed numbers. Transformations are "changes of state" such as "losing 5," "giving 3," etc., and static relationships are relationships such as: "has 8 more than," "owes 6 dollars to," etc.

For each of the six problem types above, Vergnaud pointed out that several problems can be generated depending upon which entity is to be the unknown, and if directed numbers are involved, whether positive or negative numbers are used. The problem types proliferate when each of the entities is a transformation; 18 distinct problem types are possible for the composition of two transformation categories alone.

The sections that follow are based on the analysis presented in Table 2. Although this analysis is not as complete as Vergnaud's, and does not unambiguously characterize all addition and subtraction word problems (cf. Fuson, 1979), it is useful to help clarify distinctions between problems that are accessible to young children and to help distinguish between problems with clearly different semantic characteristics that result in different methods of solution.

Strategies

In the section on Symbolic problems, a number of strategies were described that have been tested using response latencies. These strategies generally involved setting and incrementing or decrementing a mental counter. Most of the research on the strategies that children use to solve word problems have used a different research paradigm. Generally, strategies have been observed directly, using individual interviews, and children often have had manipulative objects available to assist them in their solutions. Consequently, although many of the same basic strategies have been observed, there are some important differences in how they have been characterized. In general, the descriptions of strategies based on direct observation provide more detail regarding overt manifestations of counting processes. Strategies are described in terms of modeling action or relations in problems rather than in terms of constructing counting sequences. Alternative strategies are also identified.

Early studies of solution strategies. Several early studies attempted to characterize children's addition and subtraction strategies for solving word problems. Although other researchers had directly studied strategies children use to solve symbolic problems (Brownell, 1928; Ilg & Ames, 1951), Gibb's (1953, 1956) studies represented a pioneering effort to focus on strategies used for solving various types of verbal subtraction problems. While she did not systematically report the frequency of strategies, she did categorize strategies on an ordinal scale (1-9). The nine levels of strategies were:

(1) No attempt

(2) Guess, incorrect response with no justification, or given number

(3) Counting with fingers

(4) Counting all or separating from

(5) Counting on starting with 1, e.g., "1, 2, 3, 4, 5, 6, 7"

(6) Counting on or back (without counting one of the groups)

(7) "Working with groups" without reference to counting or a number fact (not clear)

(8) Use of number fact or heuristic (derived fact)

(9) Spontaneous response (quick)—subject not able to recall the process used

Other researchers have also classified children's solution strategies on verbal addition and subtraction problems. Hebbeler (1977) identified six categories of strategy used by children in grades pre-K to 2 on addition problems presented in concrete format. Her categories are somewhat more global than Gibb's and are given below:

(1) Counting

(2) Subitizing

(3) Use of number fact

(4) Guessing

(5) Uncodable--ambiguous

(6) No attempt

Ginsburg and Russell (1979a, 1979b) used a similar classification of strategies used by 4 and 5 year olds on verbal addition problems presented in a concrete format. Their categories were:

(1) Counting

(2) Correct noncounting strategy

(3) Counting with some error (miscount)

(4) Guessing

(5) Other incorrect strategy

This categorization is again somewhat global, although preschoolers' strategies can be assumed to be somewhat more limited than those found with older children.

These classification schemes introduce some important distinctions not found in the strategies based on response latencies, but the categories fail to differentiate between strategies along certain critical dimensions that are directly related to the structure of different types of problems. The characterization below provides a more comprehensive description of children's strategies. It is based largely on the work of Carpenter and Moser (1982), but incorporates most of the distinctions found in other analyses of solution strategies. In fact, virtually all of the important distinctions have been observed in a variety of studies that will be discussed in the following section of this review.

Addition strategies. Three basic levels of addition strategies have been identified: strategies based on direct modeling with fingers or physical objects, strategies based on the use of counting sequences, and strategies based on recalled number facts. In the most basic strategy, physical objects or fingers are used to represent each of the addends, and then the union of the two sets is counted, starting with one (Counting All). Theoretically, there are two ways in which this basic strategy might be carried out. Once the two sets have been constructed, they

could by physically joined by moving them together or adding one set to the other, or the total could be counted without physically joining the sets. This distinction is important. The first case would best represent the action of the Change/Join problems while the second would best represent the static relationships implied by the Combine problems. Children generally do not distinguish between the two strategies in solving either Change/Join or Combine problems. Thus, is appears that there is a single Counting All With Models strategy. The strategy may be accompanied by different ways of organizing the physical objects, but the arrangements do not represent distinct strategies or different interpretations of addition.

A third alternative is also possible. A child could construct a set representing one addend and then increment this set by the number of elements given by the other addend without ever constructing a second set. Such a strategy would seem to best represent a unary conception of addition (Weaver, 1982). This strategy is seldom used.

There are three distinct strategies involving counting sequences. In the most elementary strategy, the counting sequence begins with one and continues until the answer is reached. This strategy, which is also a Counting All strategy, is the SUM strategy identified by Suppes and Groen (1967) and Goren and Parkman (1972). It is similar to the Counting All With Models strategy except that physical objects or fingers are not used to represent the addends. However, this strategy and the two following counting strategies require some method of keeping track of the number of counting steps that represent the second addend in order to know when to stop counting. Keeping-track procedures are

discussed in detail in Fuson (1982). Most children simultaneously count on their fingers, but a substantial number give no evidence of any physical action accompanying their counting. When counting is carried out mentally, it is difficult to determine how a child knows when to stop counting. Some children appear to use some sort of rhythmic or cadence counting such that counting words are clustered into groups of two or three. Others have explicitly described a double count, but children generally have difficulty explaining this process. When fingers are used, they appear to play a very different role than in the direct modeling strategy. In this case, the fingers do not seem to represent the second addend per se, but are used to keep track of the number of steps in the counting sequence. When using fingers, children often do not appear to have to count their fingers, but can immediately tell when they have included a given number of fingers. Steffe (personal communication) has hypothesized that finger patterns play a critical role in the development of advanced counting strategies.

The other two counting strategies are more efficient and imply a less mechanical application of counting. In applying these strategies, a child recognizes that it is not necessary to reconstruct the entire counting sequence. In Counting On From First, a child begins counting forward with the first addend in the problem. The Counting On From Larger strategy is identical except that the child begins counting forward with the larger of the two addends. This strategy is the MIN strategy of Groen and Parkman (1972).

Although learning of basic number facts appears to occur over a protracted span of time, most children ultimately solve simple addition

and subtraction problems by recall of number combinations rather than by using counting or modeling strategies. Certain number combinations are learned earlier than others; and before they have completely mastered their addition tables, many children use a small set of memorized facts to derive solutions for addition and subtraction problems involving other number combinations. These solutions usually are based on doubles or numbers whose sum is 10. For example, to solve a problem representing $6 + 8 = \square$, a child might respond that $6 + 6 = 12$ and $6 + 8$ is just 2 more than 12. In an example involving the operation $4 + 7 = \square$, the solution may involve the following analysis: $4 + 6 = 10$ and $4 + 7$ is just 1 more than 10.

Hatano (1980) identified a type of derived strategy used by Japanese children. It is a mental strategy which relies on the use of number facts other than the one directly related to the problem being solved, but it is based on the use of 5 as an intermediate unit. This mental regrouping strategy involves breaking up addends into forms such as "5 + n" where $n < 5$, e.g., for $5 + 7$ the child thinks $5 + (5 + 2)$ or $10 + 2$. Hatano cites three pieces of evidence which support the use of this strategy by Japanese children: (a) skilled abacus users rely on an internalized system based on numbers complementary to 10 and 5, (b) Counting On strategies are not observed with first and second grade Japanese children; and (c) latency data indicate that Japanese children exhibit lower latencies when addends of 5 are used. Hatano's findings suggest that some of the strategies children use on verbal addition and subtraction problems are culturally dependent rather than universal.

Subtraction strategies. Each of the three levels of abstraction described for addition strategies also exist for the solution of subtraction problems. However, whereas a single basic interpretation of addition has been the rule, a number of distinct classes of subtraction strategies have been observed at the direct modeling and counting levels.

One of the basic strategies involves a subtraction action. In this case, the larger quantity in the subtraction problem is initially represented and the smaller quantity is subsequently removed from it. When concrete objects are used, the strategy is called Separating From. The child constructs the larger given set and then takes away or separates, one at a time, a number of objects equal to the given number in the problem. Counting the set of remaining objects yields the answer. There is also a parallel strategy based on counting called Counting Down From. A child initiates a backward counting sequence beginning with the given larger number. The backward counting sequence contains as many counts as the given smaller number. The last number uttered in the counting sequence is the answer.

The Separating To strategy is similar to the Separating From strategy except that elements are removed from the larger set until the number of objects remaining is equal to the smaller number given in the problem. Counting the number of objects removed provides the answer. Similarly, the backward counting sequence in the Counting Down To strategy continues until the smaller number is reached and the number of words in the counting sequence is the solution to the problem.

The third pair of strategies involves an additive action. In an additive solution, the child starts with the smaller quantity and con-

structs the larger. With concrete objects (Adding On), the child sets out a number of objects equal to the small given number (an addend). The child then adds objects to that set one at a time until the new collection is equal to the larger given number. Counting the number of objects added on gives the answer. In the parallel counting strategy, Counting Up From Given, a child initiates a forward counting strategy beginning with the smaller given number. The sequence ends with the larger given number. Again, by keeping track of the number of counts uttered in the sequence, the child determines the answer.

The fourth basic strategy is called Matching. Matching is only feasible when concrete objects are available. The child puts out two sets of cubes, each set standing for one of the given numbers. The sets are then matched one-to-one. Counting the unmatched cubes gives the answer.

A fifth strategy, the Choice strategy, involves a combination of Counting Down From and Counting Up From Given, depending on which is the most efficient. In this case, a child decides which strategy requires the fewest number of counts and solves the problem accordingly. For example, to find 8 - 2, it would be more efficient to Count Down From whereas the counting Up From Given strategy would be more efficient for 8 - 6.

As with addition, modeling and counting strategies eventually give way to the use of either recalled number facts or derived facts. Children's explanations of their solutions suggest that the number combinations they are calling upon are often addition combinations. To explain how they found the answer 13 - 7, many children respond that they just

knew that 7 + 6 = 13. Many of the derived subtraction facts are also based on addition. For example, to explain the solution to find 14 - 8, one child reported, "7 and 7 is 14; 8 is 1 more than 7; so the answer is 6" (Carpenter, 1980b, p. 319).

Problem Structure and Solution Process

Subtraction. As can be seen from the descriptions of problem structure and children's processes, certain strategies naturally model the action described in specific subtraction problems. The Separate/Result Unknown problems (see Table, problem 2) are most clearly modeled by the Separating and Counting Down From Given strategies, whereas the Separate/Change Unknown problems (problem 4) are best modeled by the Separating To and Counting Down To strategies. On the other hand, the implied joining action of the Join/Change Unknown problem (problem 3) is most closely modeled by the Adding On and Counting Up strategies. Compare/Difference Unknown (problems 9 and 10) deal with static relationships between sets rather than action. In this case, the Matching strategy appears to provide the best model.

For the Combine subtraction problem, the situation is more ambiguous. Since Combine problems have no implied action, neither the Separating nor Adding On strategies (or their counting analogs), which involve action, exactly model the given relationship between quantities. And since one of the given entities is a subset of the other, there are no two distinct sets that can be matched.

For Equalize problem the situation is reversed. Since Equalize problems involve both a comparison and some implied action, two different

strategies would be consistent with the problem structure. The Equalize/Join problems (problem 15) involve a comparison of two quantities and a decision of how much should be joined to the smaller quantity to make them equivalent. Either the Matching or the Adding On (Counting Up From Given) strategies might be appropriate. For the Equalize/Separate problems (problem 16), the implied action involves removing elements from the larger set until the two sets are equivalent. This action seems to be best modeled by the Separating To strategy, although the Matching strategy is again appropriate for the comparison aspect of the problem.

The results of a number of studies consistently show that young children use a variety of strategies to solve different subtraction problems and that the strategies used generally tend to be consistent with the action or relationships described in the problem (Blume, 1981; Carpenter, Hiebert, & Moser, 1981; Carpenter & Moser, 1982; Hiebert, 1981). This tendency is especially pronounced for children below the second grade, but for some children the structure of the problem influences their choice of strategy at least through the third grade.

The results summarized in Table 3 are from a three-year longitudinal study of the processes that children use to solve basic addition and subtraction word problems (Carpenter & Moser, 1982). The study involved approximately 100 children who were individually interviewed, three times a year in the first and second grades, and twice in the third grade. The study included problems similar to problems 1, 2, 3, 7, 8, and 9 in Table 2. The results reported in Table 3 are for problems involving basic subtraction facts with the larger number between 11 and 16. Manipulative objectives were available to aid in the solution, but children were not required

Table 3

Relation of Strategy to Problem and Structure

(Carpenter and Moser, 1982)

Problem Type	Grade	Percent Correct	Strategy (Percent Responding)						
			Subtractive		Additive			Numerical	
			Separating From	Counting Down From	Adding On	Counting Up From Given	Matching	Recalled Fact	Derived Fact
Separate/ Result Unknown	1	61	68	1	1	3	0	1	2
	2	83	34	8	1	10	0	20	9
	3	95	9	3	1	12	0	54	13
Join/ Change Unknown	1	57	2	0	42	12	1	2	4
	2	93	1	2	18	31	0	25	16
	3	95	0	1	6	27	1	48	14
Compare/ Difference Unknown	1	41	8	0	3	9	30	1	1
	2	70	11	6	2	17	14	19	7
	3	89	3	3	2	14	2	52	17
Combine/ Part Unknown	1	45	45	0	4	3	0	2	2
	2	78	36	5	0	11	0	20	14
	3	91	6	1	0	13	0	53	18

to use them. To simplify the table, only results for the January interviews are included. At the time of the grade 1 interview, the children in the study had received no formal instruction in addition and subtraction. By the second-grade interview, they had received about six months of instruction in addition and subtraction, but mastery of number facts was not expected, and there had been no instruction on the subtraction algorithm. By the third-grade interview, students were expected to have learned their number facts and to have learned the addition and subtraction algorithms.

In grade 1, the vast majority of responses were based on problem structure. Almost all of the first graders who solved the problems correctly used the Separating From or Counting Down From strategies for the Separate problem and the Adding On or Counting Up From Given strategy for the Join/Change Unknown problem. The results were not quite so overwhelming for the Compare problem, but the Matching strategy was used by the majority of children who solved the problem correctly. Furthermore, it was the only problem for which more than two children used a matching strategy. The results for the Combine problem, for which there is no clear action to represent, generally tended to parallel those of the Separate problem.

By the second grade, about a third of the responses were based on number facts and the effect of problem structure was not quite so dominant; however, the structure of the problem continued to influence the responses of a large number of second graders. To solve the Separate problem 42% of the second graders used a subtractive strategy, while only 11% used an additive strategy. For the Join problem, 49% used an additive strategy

and only 3% used a subtractive strategy. Thus, for these two problems, most of the children who used a counting or modeling strategy continued to represent the action described in the problem. The structure of the Compare problem did not continue to exert as strong an influence, and many second graders abandoned the Matching strategy for the more efficient Separating From or Counting Up From Given strategies.

By the third grade, about two-thirds of the responses were based on number facts, and there was more flexibility in the use of counting strategies. For the Separate problem, the most popular non-numerical strategy was Counting Up From Given, and the Matching strategy was seldom used for the Compare problem. For the Join problem, however, almost all children who did not use number facts used an additive strategy. This represented about a third of the third graders in the study.

There are two plausible explanations for the continued reliance on additive strategies for the Join problem. The Counting Up From Given strategy may simply be the most efficient strategy available to some third graders for solving subtraction problems of any kind. In other words, for some children, the choice of a Counting Up From Given strategy for the Join problem may not have been dictated by the additive structure of the Join problem, it may simply be the strategy they use to solve subtraction problems. The fact that 12-14% of the third graders used the Counting Up From Given strategy for the other three subtraction problems supports this hypothesis. On the other hand, almost twice as many third graders used the additive strategy for the Join problem as for the other three problems.

The influence of the additive structure of the Join problem is apparent in the solution of two-digit problems, which are most efficiently solved using the subtraction algorithm. By the time of the January interview, the third graders in the study were expected to have mastered the subtraction algorithm. For the other five types of problems, between 80% and 95% of the children used the standard addition or subtraction algorithm; but for the Join/Change Unknown problem, only 53% used the subtraction algorithm. Almost a third used some form of additive strategy that paralleled the structure of the problem.

Results of other studies investigating the processes that children use to solve word problems are generally consistent with those of the longitudinal study reported above (Anick, in preparation; Blume, 1981; Carpenter, Hiebert, & Moser, 1981; Hiebert, 1981). These studies also included problems not administered by Carpenter and Moser (1982). Perhaps the most compelling evidence showing the effect of problem structure is found on Separate/Change Unknown problems. The strategy that best represents the action described in these problems is the Separating To strategy. In general, this strategy appears somewhat inefficient and unnatural. The strategy involves removing elements from a set until the number remaining is equal to a given value. With the Separating From or Adding On strategies, the elements that are removed or added can be sequentially counted as they are removed or added. With the Separating To strategy, however, the size of the remaining set must regularly be re-evaluated. Furthermore, the Separating To strategy is virtually never taught explicitly. In spite of these limitations, Anick (in preparation) and Hiebert (1981) found that the Separating To strategy was used by

approximately half of the children in their studies who were able to solve the Separate/Change Unknown problem. Results for the Equalize problems also generally followed the predicted pattern (Carpenter, Hiebert, & Moser, 1981).

The one case in which results were not consistent across the studies involves the use of the Matching strategy. Matching was the primary strategy for Compare problems in both the Carpenter and Moser (1982) study and in the Carpenter, Hiebert, and Moser (1981) study. It was almost never found, however, in the studies of Anick (in preparation) or Riley et al. (in press). This may be due to the type of mathematics program used by the children studied. In the studies where Matching was used, the mathematics program in use was Developing Mathmematical Processes (DMP) (Romberg, Harvey, Moser, & Montgomery, 1974), a program which provides early experience in comparing the relative size of two sets by matching. It appears that children who have used Matching to compare sets can extend this process to find the magnitude of the difference in Compare problems without explicit instruction. If children have no experience matching sets, they do not spontaneously apply the process to Compare problems, in which case, they have no way to represent the relationship described in the Compare problems. As a consequence, these problems are very difficult for them (Anick, in preparation; Riley et al., in press).

These results clearly illustrate the importance of examining children's solution processes. Although there were significant differences in the success level for Compare problems between these two sets of studies, the results of both are generally consistent with the

conclusion that children's earliest solution processes are based on modeling the action or relationships described in the problem. However, in one instance, children had available to them a process with which to model the relationship described in the Compare problems; in the other, they did not.

The results for the Compare problems demonstrate that if children do not have a process available to model the action or relationships in a given problem, the problem is much more difficult that if it can be directly modeled. Certain types of problems are difficult to model. For example, in the Change/Start Unknown problems (Table 2, problems 5 and 6), the initial quantity operated on to yield a given result is unknown. To directly model the action in a Change problem requires that there is an initial set to either increase or decrease. Therefore, to model the action in the Start Unknown problems would require some sort of trial and error in which one guessed at the size of the initial set and then performed the specified transformation to check whether it produced the given result. Rosenthal and Resnick (1974) hypothesized that children may use such a process. But recent data from Anick (in preparation) indicate that trial and error is almost never used. Consequently, Start Unknown problems should be significantly more difficult than problems that can be modeled directly.

An analysis of the relative difficulty of different types of word problems indicates that problems that cannot be easily modeled are significantly more difficult than those that can. The results of a study by Riley et al. (in press) are summarized in Table 4. The Start Unknown problems were found to be significantly more difficult than the other

Table 4

Relative Difficulty of Word Problems

(Riley, Greeno, and Heller, in press)

Problem Type	Grade			
	K	1	2	3
Join/Result Unknown (1)[a]	.87[b]	1.00	1.00	1.00
Separate/Result Unknown (2)	1.00	1.00	1.00	1.00
Join/Change Unknown (3)	.61	.56	1.00	1.00
Separate/Change Unknown (4)	.91	.78	1.00	1.00
Join/Start Unknown (5)	.09	.28	.80	.95
Separate/Start Unknown (6)	.22	.39	.70	.80
Combine/Addition (7)	1.00	1.00	1.00	1.00
Combine/Subtraction (8)	.22	.39	.70	1.00
Compare/Difference Unknown (9)	.17	.28	.85	1.00
Compare/Difference Unknown (10)	.04	.22	.75	1.00
Compare/Compared Quantity Unknown (11)	.13	.17	.80	1.00
Compare/Compared Quantity Unknown (14)	.17	.28	.90	.95
Compare/Referent Unknown (13)	.17	.11	.65	.75
Compare/Referent Unknown (12)	.00	.06	.35	.75

[a]Problem number in Table 2.

[b]Proportion of children responding correctly.

four change problems. Since subjects in the study had not been exposed to the Matching strategy, all of the Compare problems were difficult for them. The other problem that proved to be especially difficult was the Combine/Part Unknown problem. Since no clear action can be modeled in the Combine problem, these results are also consistent with the hypothesized effect of problem structure.

Differences in the wording of problems, experimental protocols and procedures, and student backgrounds all complicate the comparison of different studies that examine problem difficulty. However, although most studies have not found the clear differences reported by Riley et al. (in press), other studies do support the conclusion that Start Unknown problems are more difficult than other Change problems (Anick, in preparation; Lindvall & Ibarra, 1979), and that Compare and Combine subtraction problems are relatively difficult to solve (Anick, in preparation; Gibb, 1956; Nesher, 1982; Schell & Burns, 1962; Shores & Underhill, 1976). There is also relatively consistent evidence that Join/Change Unknown problems are more difficult than Separate problems (Gibb, 1956; Hirstein, 1979; Lindvall & Ibarra, 1979; Rosenthal & Resnick, 1974; Schell & Burns, 1962; Shores & Underhill, 1976; Steffe et al., 1976).

Addition. Whereas children have multiple conceptions of subtraction, reflected in the different processes used to solve different problems, they appear to have a reasonably unified concept of addition. Children appear to treat Join and Combine addition problems as though they were equivalent. Not only are the same basic processes used for both problems, but the same pattern of responses appears for both. The similarity of responses for the two types of problems is illustrated by

the results from Carpenter and Moser (1982), summarized in Table 5. These results are from the September interviews of the same children reported on in Table 3, this time involving basic addition problems with sums between 11 and 16 in which manipulative objects were not available.

Studies of problem difficulty support the conclusion that there is little difference in children's solutions to Join and Combine addition problems (Grunau, 1978; Lindvall & Ibarra, 1979; Nesher, 1982; Shores & Underhill, 1976; Steffe, 1970; Steffe & Johnson, 1971). In most cases, performance was not markedly different on these two types of items. At the kindergarten level, three of these studies (Grunau, Lindvall and Ibarra, and Shores and Underhill) reported slightly better performance on the Combine problem than on the Join (p values were .46 and .59, respectively, for Join and Combine problems in Shores and Underhill; .55 and .63 in Grunau; and .47 and .54 in Lindvall and Ibarra). At the first-grade level, Shores and Underhill and Steffe and Johnson found nearly identical performance on the two types of addition verbal problems. On problems with sums less than 10, p values ranged from .63 to .88 across these studies, indicating that first graders do well with both kinds of addition problems. Steffe reported p values of .85 and .77 for Join and Combine problems with first graders. Nesher found similar results at the elementary level with p values of .89 and .75 for Join and Combine problems.

Differences in performance do exist, however, between Join and Combine addition problems and some of the other categories of addition word problems. In fact, the Separate/Start Unknown and Compare addition problems are significantly more difficult than Join or Combine problems (see Table 4), because the structure of the Separate and Compare problems

Table 5

Combine and Join Addition Problems

(Carpenter and Moser, 1982)

Grade	Problem Type	Percent Correct	Strategy (Percent Responding)				
			Counting All	Counting On From First	Counting On From Larger	Derived Fact	Recalled Fact
1	Combine	50	52	3	3	1	1
	Join	47	46	3	8	2	1
2	Combine	72	39	6	29	4	7
	Join	84	41	14	26	6	6
3	Combine	91	13	7	33	11	30
	Join	90	11	15	32	9	32

are not directly modeled by the addition strategies children have available. This lack of congruence between problem structure and available solution strategies is not there in Join and Combine addition problems for which the Counting All and Counting On strategies provide reasonable models.

Children appear to use the same basic processes to solve all types of addition problems (Carpenter, Hiebert, & Moser, 1981); however, problem structure also influences children's solutions. The problems that can be reasonably modeled by available strategies are relatively easy whereas those that cannot are significantly more difficult.

DEVELOPMENT OF ADDITION AND SUBTRACTION PROCESSES

Much of the early research on addition and subtraction focused on factors affecting problem difficulty. This approach provided a relatively static view of children's performance. Certain problems were identified as being more difficult than others, but generally no attempt was made to describe the different levels of difficulty in terms of a developmental hierarchy, i.e., in which the acquisition of ability to solve more difficult problems built upon the abilities used to solve the easier problems.

A primary focus of current research is to describe how addition and subtraction concepts are acquired by children over time. This emphasis reflects the influence of Piaget; however, while Piaget's development model focuses on children's limitations and misconceptions at early stages of development, much of the current research describes a sequence of development for addition and subtraction concepts wherein perfectly valid solution processes are replaced by increasingly efficient and abstract processes.

Reasonably clear evidence has been found which indicates that there is a level at which children can only solve addition and subtraction problems by directly modeling with physical objects or fingers the action or relationships described in the problem. At this level, children cannot solve problems that cannot be directly modeled. Results for the lowest grade levels reported in Tables 3, 4, and 5 provide compelling evidence for the existence of this level in children's acquisition of addition and subtraction concepts and skills. The great majority of the youngest children's responses were limited to direct modeling strategies (Tables 3 and 5), and few of the younger children could solve any problem that could not be directly modeled (Table 4).

Development of the more advanced levels of children's solution processes proceeds along two dimensions: an increase in the level of abstraction and an increase in the flexibility of strategy choices.

Level of Abstraction

Direct modeling. At the most primitive level of solving addition and subtraction problems, children completely model the action or relationships in the problem using physical objects or fingers. They actually construct sets to represent all of the quantities described in the problems. The Counting All strategy is the addition strategy used at this level. The parallel subtraction strategies are Separating From, Separating To, Adding On, and Matching.

Counting sequences. At the next level, the external direct modeling actions of this initial concrete level become internalized and abstracted, allowing greater flexibility and efficiency. Children no longer have to physically represent each of the quantities described in

the problem. Instead they are able to focus on the counting sequence itself. They realize that they do not have to actually construct the sets or even go through the complete counting sequence to find the number of elements in the union of two sets. They can start at a number representing one quantity and count on the number representing the other quantity. Although children do use finger patterns in conjunction with the Counting On strategy, fingers are used in a very different sense than at the direct modeling level. Children's explanations of their use of Counting On suggest that these finger patterns do not represent the second set per se but are simply a tally of the number of steps counted on. The abstraction of the subtraction strategies involves essentially the same basic pattern of development, except that in the case of the Separating strategies, a backward counting sequence is required.

The shift from complete modeling to use of counting strategies depends upon the development of certain basic number concepts and counting skills. Fuson (1982) has argued that counting on depends upon understanding basic principles which involve (a) the relation between cardinality and counting and (b) recognizing that each addend plays a double role as both an addend and a part of the sum. The counting skills required include (a) the ability to begin a counting sequence at any number, (b) the ability to maintain a doubles count, and (c) in the case of subtraction, the ability to count backwards.

The role of these concepts and skills in the development of the Counting On strategy is discussed in detail in Fuson (1982). The essential point is that counting strategies are not simply mechanical techniques that children have learned to solve addition and subtraction problems, but are conceptually based strategies which directly build

upon the direct modeling strategies of the previous level. The abstraction and flexibility demonstrated by their application imply a deeper understanding of number and addition and subtraction than was found at the direct modeling level, but this understanding is based upon the conceptualizations of the initial concrete actions. In other words, the levels are not independent; they build on one another.

There is a clear parallel between the development of counting strategies and the stages of development described by Piaget. In both cases, operations initially performed with external concrete actions are internalized, providing for greater flexibility in their application.

Children continue to use counting strategies for an extended period of time. Lankford (1974) found that as many as 36% of the seventh graders he interviewed continued to use counting strategies to arrive at some basic addition and subtraction facts. Furthermore, children become so proficient and quick in the use of the counting strategies and so covert in their use of fingers as tallying devices that it is often difficult to distinguish between the use of a counting strategy and recall of a basic number fact. However, relying on counting strategies when solving more complex problems that require algorithms is inefficient and provides too much opportunity for miscalculation. Learning number facts at a recall level remains a viable goal of the mathematics curriculum, and most students eventually attain this level.

Use of number facts. There is no clearly distinct shift from counting strategies to use of number facts. As the research on difficulty of different number facts indicated, some facts are learned and used earlier than others; and there is a long period when children use a

combination of number facts in conjunction with direct modeling of counting strategies.

Not a great deal is known about how the use of counting strategies evolves into or affects the larning of basic facts at the recall level. Leutzinger (1979) investigated the effect of Counting On on the learning of basic addition facts, and other studies have provided instruction on a number of types of strategies, including counting strategies, to provide some structure to facilitate recall of basic facts (Rathmell, 1979; Swenson, 1949; Thiele, 1938; Thornton, 1978). Aside from finding that such instruction has generally proved effective, it is still not clear exactly how recall of number facts related to children's counting strategies.

The relation between counting strategies and learning basic facts at the recall level is one issue. A second is the relation between different facts. As noted earlier, children occasionally use known facts to derive facts that they do not know at the recall level. Carpenter (1980b) proposed that the use of derived strategies is not limited to a select group of superior students. By the end of first grade, over half of the students in the study reported by Carpenter and Moser (1982) had used a derived strategy at least once, and by the middle of second grade, over three-fourths of the students had done so. Children who used derived strategies, however, did not use them consistently. For the smaller numbers, only one child used a derived strategy more than three times in the 12 problems administered. For the larger numbers, only one first grader used more than three derived strategies. Four second graders used more than three derived strategies at the September interview and 12 did so in January.

It is tempting to assume that derived strategies are used during a transitional level between the use of counting strategies and the routinization of number facts. The data from Carpenter and Moser (1982) suggested, however, that things are not that simple. So far, we have been able to establish no clear connection between the use of derived strategies and the levels of development of either modeling or counting strategies. Derived strategies are occasionally used by children using the most primitive modeling and counting strategies.

Some studies have claimed success for explicit instruction in strategies that can be used to derive unknown facts from known facts (Swenson, 1949; Thiele, 1938; Thornton, 1978). But the role that derived strategies play in the learning of basic facts at the recall level is far from clear (Rathmell, 1979; Steffe, 1979). Potentially, these strategies provide a logical basis for relating facts that could facilitate recall. Furthermore, Brownell (1928) strongly argued that students must be capable of the reasoning involved in derived strategies in order to give meaning to memorized addition and subtraction combinations.

Generalizability of levels of abstraction. One of the questions regarding the development of different levels of abstraction is whether the use of more advanced strategies is broadly based across all problems or whether children use advanced strategies on some problems but not on others. Results of Carpenter and Moser (1982) suggested that children use different patterns of increased abstraction for different strategies and problems. The Counting Up From Given strategy is used much earlier and more frequently than the Counting Down From strategies. For the Separating subtraction problem, no more than 15% of the children used the

Counting Down From strategy in any interview. The Counting Up From Given strategy, on the other hand, accounted for as many as 50% of the responses to the Join/Change Unknown problem. In fact, it appears that some children may never use a Counting Down From strategy.

There does seem to be a relation between the use of Counting On strategies for addition and the Counting Up From Given strategy for subtraction. Many children appear to see them as essentially the same strategy. A number of children when asked to explain their use of the Counting Up From Given strategy said that they did the same thing that they had done on the previous problem, which was an addition problem that they solved by counting on.

Choice of Strategy

The second dimension along which development occurs is in the flexibility in choice of strategy. At first, the only problems that young children can solve are those for which the action or relationships described in the problem can be directly modeled. By the second or third grade, however, many children are able to use strategies that are not entirely consistent with the structure of the problem (see Tables 3 and 4).

As with the shift to more abstract counting strategies, children's flexibility in choice of strategy is not consistent over problems. Children soon abandoned the somewhat complicated Matching strategy for the Compare problem, but were much less flexible in their choice of strategy for the Separate and Join problems (see Table 3). In fact, even though children were much more successful in applying Counting Up

From Given than they were with Counting Down From strategies, fewer than 15% ever used Counting Up From Given to solve Separating From problems.

Flexibility in the choice of strategy also affects children's ability to solve numerical open sentence problems. Young children tend to use a Separating From strategy to solve problems of the form a - b = ☐ and the Counting Up From Given strategy to solve problems of the form a + ☐ = c (Blume, 1981). Woods et al. (1975) and Groen and Poll (1973) suggested that the increasing flexibility to choose between strategies is also reflected in the choice of strategy for solving numerical subtraction problems. They presented response latency data that they argue is best explained by the Choice strategy, which involves choosing either a Counting Up From Given or Counting Down From strategy, depending on which requires fewer steps. The support for the widespread use of such a strategy has not been uniformly consistent. Although it does seem to present the best fit with the data from response latency studies, Blume (1981) found little evidence of such a strategy, using clinical interviews. Intuitively, it would seem that children would use such a strategy with problems with only a few steps involved (e.g., 11 - 3, or 11 - 9), but would be less likely in situations where the choice was not as clear (e.g., 12 - 5). One also has to be cautious in interpreting the data in support of the Choice strategy. Some of the children hypothesized to be using the Choice strategy were in the fourth grade. Most children of this age have memorized the basic facts used in these studies. Consequently, the latency data may reflect something other than the overt use of counting strategies.

To have a completely developed concept of subtraction, children should recognize the equivalence of the different strategies. This knowledge should make the Choice strategy possible. But the evidence suggests that many children avoid Counting Down From strategies. Consequently, some caution should be exercised in assuming widespread use of the Choice strategy over a broad class of problems.

Relationship Between Dimensions of Development

The previous section of this paper described the increase in flexibility and abstraction that occurs over time in children's processes for solving addition and subtraction problems. An important question to ask is whether certain levels of abstraction require a more flexible choice of strategy or vice versa. Attempting to characterize the relationship between these two dimensions into which children's strategies evolve is complicated by the fact that children do not consistently use their optimal strategies. For example, throughout first grade, children in Carpenter and Moser's (1982) study solved Join/Change Unknown problems using counting strategies rather than direct modeling strategies almost twice as often when cubes were not available as when they were. In fact, direct modeling strategies were used more frequently for problems with smaller numbers than for problems with larger numbers.

It appears that there is a shift in the level of abstraction of children's solution processes before children begin to recognize the equivalence of different subtraction strategies. In other words, the first evidence of growth is that children begin to use counting strategies that parallel the action in a problem instead of completely modeling

the action. The results summarized in Table 6 are taken from the June interviews. Number combinations involved the difference of a two-digit and a one-digit number fact, and manipulatives were not provided. Almost half of the first-grade children used a counting strategy rather than completely modeled the problem. On the other hand, most children used either a modeling or counting strategy that directly represented the action in the problem.

The relationship between children's understanding of the equivalence of different additional and subtraction strategies and their ability to use either derived or recalled number facts is a bit more difficult to establish. Many derived number facts are based on understanding relationships between addition and subtraction and suggest an understanding of the equivalence of different subtraction strategies. Similarly, for many children, recall of subtraction number facts is based on their knowledge of addition facts. On the other hand, many children learn some number facts and generate derived facts before they give any evidence of being able to use modeling or counting strategies that are not consistent with the structure of the problem.

In fact, the ability of some children to solve problems that cannot be directly modeled may be based on their ability to relate the problem to known number facts rather than to an understanding of the relationships in the problem that would allow them to choose different counting or modeling strategies. Many children do not solve problems that cannot be readily modeled until they would be expected to have learned the related number facts. However, although Anick (in preparation) found a high incidence of recall in children's solutions to problems that could not

Table 6

Results for Selected Separate and Join Subtraction Problems

(Carpenter and Moser, 1982)

Problem Type	Percent Correct	Strategy (Percent Responding)				
		Subtractive		Additive		
		Separating	Counting Down From	Adding On	Counting Up From Given	Derived/Recalled Number Fact
Separate/Result Unknown	45	23	14	1	8	15
Join/Change Unknown	61	1	0	14	43	13

be directly modeled, she also found many responses that were based on counting strategies. This suggests that even the problems that cannot be directly modeled are not just solved at an abstract numerical level. Children can solve these problems by understanding how the action or relationships in the problems are related to the counting processes that represent the different actions or relationships. How this may occur is the topic of the next section of this paper.

A Model of Verbal Problem Solving

Several models have been proposed which describe stages for solving addition and subtraction problems and which hypothesize knowledge structures sufficient to account for the behavior in these stages. Briars and Larkin (unpublished) and Riley et al. (in press) have developed computer simulation models to solve addition and subtraction word problems that provide a very similar analysis of the basic knowledge at each stage of problem solving.

Riley et al. identified three basic kinds of knowledge involved in problem solving: (a) problem schemata which are used to represent the problem situation, (b) action schemata which, at the most global level, essentially correspond to the solution processes described earlier, and (c) strategic knowledge for planning solutions to problems.

Based on the results summarized in Table 4, Riley et al. identified the levels of skill for solving Change, Combine, and Compare problems, and a computer simulation model was constructed for each level.

For Change problems, Model 1 is limited to external representations of problem situations using physical objects. Model 1 relies on Counting

All and Separating strategies. It cannot even use the Adding On strategy, since it has no way to keep track of set subset relationships.

The major advance of Model 2 over Model 1 is that it includes a schema which makes it possible to keep a mental record of the role of each piece of data in the problem. This allows Model 2 to solve Change Unknown problems (see Table 2, problem 3). Model 2 is also able to use Counting On procedures. Model 2 is limited to direct representation of problem action and is unable to solve Start Unknown problems (see Table 2, problems 5 and 6) because it is unable to represent the initial set.

Both Model 1 and Model 2 are limited to direct representation of problem structure. Model 3 includes a schema for representing part-whole relations that allows it to proceed in a top-down direction so as to construct a representation of the relationships between all the pieces of information in the problem before solving it. This frees the model from relying on solutions that directly represent the action in the problem. Model 3 can solve all six Change problems.

Similar models have been proposed for Combine and Compare problems, although computer simulations have not yet been implemented (Riley et al., in press).

The simulation models developed by Riley et al. and Briars and Larkin generate solutions to problems that are generally consistent with the way in which children solve problems. That is, their predicted patterns of both problem difficulty and solution strategies match the results of empirical studies of addition and subtraction. This suggests that the constructs upon which the models are based are sufficient to explain how children at different levels solve addition and subtraction problems.

Specifically, it implies that a part-whole schema is sufficient to account for children's ability to solve the more difficult addition and subtraction problems.

The fact that the models are sufficient to explain children's behavior does not mean that children necessarily use part-whole relationships to solve the more difficult addition and subtraction problems. Alternative explanations may also account for children's performance. For example, solutions to Start Unknown problems may be based on an understanding of the inverse relationship between addition and subtraction. So far there has been no attempt to systematically generate and test alternative models of children's behavior. Until that has been done, some caution should be exercised in drawing any firm conclusions about the mental operations involved in children's solutions. At this time, some sort of part-whole schema appears to be one of the most plausible explanations, but perhaps not the only one. That an understanding of part-whole relationships can provide a basis for solving a wide range of addition and subtraction problems is supported by the fact that explicit instruction on ways of representing part-whole relationships has met with some success in teaching addition and subtraction problem-solving skills (Kouba & Moser, 1979).

It should also be observed that the simulation models are only a reasonable first approximation for representing children's behavior. There is a great deal that they either oversimplify or do not explain. For example, the models are limited to operations on sets and fail to take into account children's knowledge of number facts. There is also a great deal less uniformity in children's behavior than is implied by

the models. Children are not consistent in their use of strategies (Blume, 1981; Carpenter, Hiebert, & Moser, 1981). Siegler and Robinson (1981) have argued that it is not sufficient to build a model of how children may apply a particular strategy; it is also necessary to acount for how they choose between alternative strategies.

DISCUSSION

Research on how children solve basic addition and subtraction problems has come a long way in the last few years. A framework for characterizing problems has evolved that helps to understand how children solve different problems and why certain problems are more difficult than others. The strategies that children use to solve addition and subtraction problems have been clearly documented in such a way as to make it possible to identify major stages in the acquisition of addition and subtraction, especially at the early levels. Very recently, models have been constructed that go a long way in characterizing internal cognitive processes which may account for children's behavior.

There is, however, a great deal that is yet unknown about how addition and subtraction concepts and skills are acquired. One of the basic assumptions of much of the research and theory-building in the area is that the processes children use to solve an addition or subtraction problem are intrinsically related to the structure of the problem. There is support for this assumption in that generally consistent results have been reported over a variety of instructional programs and within a number of different population groups.

The effects of instruction are still unclear, however, especially at the more advanced levels of children's acquisition of addition and subtraction. The specific strategies that children use may be influenced by instruction as children's use of the Matching strategy clearly shows. Hatano (1982) suggested that Japanese children may not rely predominantly on the counting strategies that have been so prevalent in American research.

There is also very little known about the transition from the informal modeling and counting strategies that children appear to invent themselves to the formal algorithms and memorized number facts that children learn as part of the mathematics curriculum. Some evidence suggests that at first children do not see any connection between their informal modeling and counting strategies and many of the formal skills they learn in their mathematics classes (Carpenter, Moser, & Hiebert, 1981). How or whether this connection is made is an important issue that so far has received relatively little attention in the growing body of research on addition and subtraction.

There is clear evidence that young children's responses to addition and subtraction problems is based on the semantic structure of the problem, but little is known regarding how children extract the meaning from the particular wording of different problems. Children's solutions clearly are not based exclusively on semantic structure.

Several recent investigations demonstrate the effect on performance of differences in wording of problems with the same semantic structure. In a study of kindergarten and first-grade children, Hudson (1980) produced significant differences in performance on a basic subtraction problem which asked children to compare the number of birds in a picture to the number of worms. In one case, children were asked how many more birds there were than worms. The problem was significantly easier, however, when children were asked the following question: "Each of these birds wants to eat a worm. How many of them will not get a worm?" Similar differences have been found for different wordings of Join/Change Unknown problems (Carpenter, Hiebert, & Moser, 1981; Riley et al., in press).

These studies indicate that the semantic structure of addition and subtraction problems does not completely determine children's performance. It is also necessary to be cautious in drawing conclusions about children's processing from specific studies, especially when difficulty level is used as the criterion measure. The fact that alternative versions of problems with the same semantic structure produce significant differences in performance does not threaten the general conclusions regarding young children's attention to problem structure. Although the difficulty levels are affected by changes in wording or syntax, the processes that children use remain relatively consistent. It appears that some wordings make the semantic structure of problems more transparent than others, but beyond that the processes used to extract meaning from the verbal statements of the problems remain something of a black box, which has not been the focus in most of the current research and theory.

Another limitation of most current research on addition and subtraction is that it does not deal with the question of individual differences. Most of the theory at least tacitly assumes that children go through essentially the same stages in acquiring addition and subtraction concepts and skills. Clearly, there are differences between children within a given grade, but generally these differences have been attributed to individual children being at different levels in acquiring basic addition and subtraction concepts and skills. So far, this assumption has not been seriously examined, and little is known about whether there are fundamentally different methods that individual children use to acquire addition and subtraction concepts and skills.

One of the reasons for the recent progress in understanding how children learn to add and subtract is the clear focus of much of the current research on a well defined domain. This research chould be criticized, however, for being too narrowly focused. Siegler and Robinson (1981) argued for the importance of building large scale, integrative models that specify how performance in addition and subtraction is related to performance in other basic content domains. They provided one example of such a model. Fuson (1982) and Steffe, Thompson, and Richards (1982) have also been attempting to examine addition and subtraction within a larger context. They have focused on the relation between addition and subtraction and the development of counting skills.

It has also been proposed that the development of specific addition and subtraction processes may depend on the development of central information processing capacities (Case, 1982). Research by Romberg and Collis (1980) provided some support for this conclusion, but a great deal must still be done to really understand how information processing capacity affects the addition and subtraction processes children are capable of using.

There is certainly a great deal left to be explained about how children learn to add and subtract. However, these details are insignificant compared to the disparity between what is already known about how children solve addition and subtraction problems and current programs of instruction (Carpenter, 1981). There is a compelling need for research that attempts to establish how this knowledge already accumulated can be applied to design instruction.

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